Synthesis of Rayleigh-wave envelope on the spherical Earth: Analytic solution of the single isotropic-scattering model for a circular source radiation

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[1] Surface waves propagating around the spherical Earth are scattered by medium heterogeneities and topographical variations. Wave trains appearing between the multiple arrivals of surface waves are interpreted as scattered waves. Mean-Squared (MS) envelopes of vertical-component seismograms in long periods can be explained by the single isotropic scattering process of Rayleigh waves of the fundamental mode based on the radiative transfer theory on the spherical Earth. For the case of circular radiation from a point source on the sphere, we can analytically solve the single scattering approximation of the radiative transfer equation. The resultant MS envelope is written by using an elliptic integral of the first kind. INDEX TERMS: 3210 Mathematical Geophysics: Modeling; 3230 Mathematical Geophysics: Numerical solutions; 7218 Seismology: Lithosphere and upper mantle; 7255 Seismology: Surface waves and free oscillations; 7260 Seismology: Theory and modeling. Citation: Maeda, T., H. Sato, and M. Ohtake, Synthesis of Rayleigh-wave envelope on the spherical earth: Analytic solution of the single isotropic-scattering model for a circular source radiation, Geophys. Res. Lett., 30(6), 1286, doi:10.1029/2002GL016629, 2003.

1. Introduction

[2] On long period seismograms, wave trains of significant amplitude are observed between the multiple arrivals of direct surface-waves propagating along the circumference of the spherical Earth. They are interpreted as surface waves scattered by distributed medium heterogeneities, anisotropic structures and topographic variations. Most of the studies about scattering of surface waves attempted to formulate scattered wave field in terms of lateral heterogeneities or vertical discontinuities based on the Born approximation and its extensions [Snieder, 1986; Friederich, 1999], and to invert for the heterogeneous structure by the analysis of waveforms [Snieder, 1988; Marquering et al., 1996]. Different from the above deterministic approaches, stochastic approaches have also been attempted to model the surface wave propagation through inhomogeneous media that have random structures. Brandenburg and Snieder [1989] calculated the scattering loss of surface waves in random media that are characterized by an exponential or a Gaussian autocorrelation function. Park and Odom [1999] studied multiple scattering processes in a medium that has random rough interfaces. For body waves the radiative transfer theory was developed to describe phenomenologically scattering process of seismic waves [e.g. Sato and Fehler, 1998]. This theory successfully describes the mean-squared (MS) envelopes of body waves at high frequencies. However, this theory has not been applied to scattered surface waves except for a few studies [Aki, 1969; Kopnichev, 1975]. We note that MS envelope corresponds to the energy density of seismic waves in the radiative transfer theory.

[3] Recently, Sato and Nohechi [2001] focused on the envelope of vertical-component seismograms in long-periods. They successfully explained observed MS envelopes for a wide lapse time range up to 30,000 s from the earthquake origin time by single isotropic-scattering of Rayleigh waves of the fundamental mode. Later, correctly formulating the multiple isotropic scattering process, Sato and Nishino [2002] solved the radiative transfer equation for Rayleigh waves on a spherical Earth. They showed that the single scattering term is enough to quantitatively explain observed 80-180s period Rayleigh-wave envelopes for the lapse time up to 30,000s, because of low value of scattering coefficient, g0 ≈ 2 × 10^{-6} km^{-1}.

[4] In the synthesis of MS envelopes on the spherical Earth, they used the numerical integration with respect to the radiation angle at the source. In the present study, we present an analytical solution of the integral for the case of circular radiation from a point source.

2. Single Isotropic Scattering Model of Surface-Waves on the Spherical Earth

[5] We assume that the Earth is a sphere of radius R, and use the spherical coordinate system (R, θ, φ) for describing a location on the Earth’s surface. We mathematically model distributed medium heterogeneities and topographic variations by a random and uniform distribution of point-like scatterers on the surface of the spherical Earth. By using scattering cross-section of each scatterer σ₀ and the number density n, the scattering power per unit area on the spherical surface is given by the total scattering coefficient g₀ ≡ nσ₀, which is the reciprocal of the mean free path. Random distribution of scatterers justifies the use of the radiative transfer theory since the scattered waves are incoherent and their power is additive. Total attenuation Q^{-1} of surface waves for a given angular frequency ω is given by the addition of scattering loss Q^{-1} = g₀V/ω and intrinsic loss Q^{-1} = g₁V/ω, where V is group velocity, and g₁ has a dimension of the reciprocal of length. We derive the single-scattering energy density of surface waves according to Sato and Nohechi [2001] as follows.
Figure 1. Geometry of the single scattering process of surface waves on a spherical Earth of radius $R$. A circular radiation source is put at the North Pole. Gray and black thick lines represent the direct path and the scattered ray path and a thin black curve is an isochronal scattering curve for normalized lapse time $t$, summing up the contributions of single scattering energy density from scattering points lying on the isochronal scattering curve, we have single-scattered energy density at the receiver as

$$E^s(t) = \frac{W}{2\pi R |\sin \theta|} \delta \left( t - \frac{R0}{V} \right) e^{-\left( \frac{p - p0}{V} \right) t}.$$

We note that the variable region of angular distance $\theta$ is $[0, \infty]$. This means that we allow a travel distance much longer than the circumference of the sphere.

A seismic ray radiated from the source at time zero has an energy flux density $VE^0(\theta', t')$ at a scatterer of angular distance $\theta'$ at time $t'$. One of the scattered rays arrives at the receiver after traveling angular distance $\theta''$ from the scatterer at $\theta'$. At a given lapse time $t$, the scattering energy density for this ray path at the receiver is

$$\frac{e^{-\left( \frac{p - p0}{V} \right) t} \delta \left( V0 - V1' - R0'' \right) g0 VE^0(\theta', t')}{2\pi R |\sin \theta'|}$$

for $0 < \theta' < \infty, 0 < \theta'' < \infty$.

At a given lapse time $t$, there are many rays that have scattered from different scattering points. We can define a specific curve on which such scattering points lie. We call this curve the isochronal scattering curve (see Figure 1). For a given lapse time $t$, summing up the contributions of single scattering energy density from scattering points lying on the isochronal scattering curve, we have single-scattered energy density at the receiver as

$$E^s(\Delta_0, t) = \int_0^{2\pi} d\phi' \int_0^\infty R^2 \sin \theta' d\theta' \int_0^\infty dt' e^{-\left( \frac{p - p0}{V} \right) t'} \frac{\delta \left( V0 - V1' - R0'' \right) g0 VE^0(\theta', t')}{2\pi R |\sin \theta'|}.$$

Substituting (1) into (3), we get

$$E^s(\Delta_0, t) = \frac{Wg0}{4\pi^2 R e^{-\left( \frac{p - p0}{V} \right) t}} \int_0^{2\pi} d\phi' \int_0^\infty d\theta' \int_0^\infty \frac{1}{|\sin \theta'|} \delta \left( \theta' + \theta'' - \tau \right).$$

where $\tau = \frac{V0}{R}$ is normalized lapse time. This representation is a special case of circular source radiation (see equation (3) in Sato and Nohechi [2001]).

### 3. Integration Using Spherical Trigonometry

In the case of angles $\theta'$ and $\theta''$ are smaller than $2\pi$, $\theta'$, $\theta''$ and $\Delta_0$ forms a triangle on the sphere. According to spherical trigonometry, angle $\theta''$ satisfies

$$\cos \theta'' = \cos \Delta_0 \cos \theta' + \sin \Delta_0 \sin \theta' \cos \phi'.$$

We note that the above relation is valid even for large values $\theta'$ and $\phi'$ exceeding $2\pi$. Considering that $\theta''$ is a function of $\theta'$ and $\phi'$, we first perform the integration of the $\delta$-function in (4) with respect to $\phi'$. The result is

$$\int_0^{2\pi} \frac{\delta \left( \theta' + \theta'' - \tau \right)}{|\sin \theta'|} d\phi' = \sum\left\{ \int_0^{2\pi} \frac{\delta \left( \theta' - \phi'' \right) d\phi'}{|\sin \theta'|} \right\}_{\phi'' = \tau - \theta'} = \frac{2}{|\sin \theta'|} \frac{d\theta'}{d\phi''}.$$

where $\phi''$ are angles that satisfy the relation $\theta'' = \tau - \theta'$. Coefficient 2 in the last term comes from the duality of this angle on both sides of the meridian connecting the source and the receiver. Substituting (5) and its derivative with respect $\phi''$ into (6), we get

$$\left( \sin \theta' \frac{d\theta''}{d\phi''} \right)_{\phi'' = \tau - \theta'} = \sin \Delta_0 \sin \theta' \sin \phi'' = \sin \Delta_0 \sin \theta' \left[ \pm \sqrt{1 - \cos^2 \phi''} \right] = \pm \sin \Delta_0 \sin \theta' \sqrt{1 - \left( \frac{\cos \left( \tau - \theta' \right) - \cos \Delta_0 \cos \theta' \tau \right)^2}{\sin \Delta_0 \sin \theta'}.}

$$\frac{\tau + \Delta_0}{2} \sin \frac{\tau - \Delta_0}{2} \sin^2 \frac{\Delta_0}{2} \left[ 1 - \csc^2 \frac{\Delta_0}{2} \sin^2 \left( \theta' - \tau \right) \right].$$

The right hand side of the first line in (7) is obviously real, so the inside of the root of the last line should be positive:

$$\frac{\tau + \Delta_0}{2} \sin \frac{\tau - \Delta_0}{2} \sin^2 \frac{\Delta_0}{2} \left[ 1 - \csc^2 \frac{\Delta_0}{2} \sin^2 \left( \theta' - \tau \right) \right] > 0.$$
The condition (8) restricts the variable range of combinations of \( \tau, \Delta_0 \) and \( \theta' \).

[9] The integral region with respect to \( \theta' \) in (4) is originally \([0, \infty] \) but because of the conditions \( \theta' = \theta - \theta' \) and \( \theta' > 0 \), the actual integral region are limited to \([0, \tau] \).

Setting \( \xi = \theta' - \xi \), the energy density of single scattering waves can be represented as

\[
E^1(\Delta_0, \iota) = \frac{W_{00}}{4\pi^2 R} e^{-\langle \varrho_0 + \varrho_0 \rangle R/2} I, \tag{9}
\]

where

\[
I \equiv \int_0^{\pi/2} d\xi \frac{\sin^2(\frac{\xi}{2}) \sin(\frac{\xi}{2})}{\sqrt{\sin^2(\frac{\Delta_0}{2}) \sin^2(\frac{\Delta_0}{2})}} \left(1 - \frac{1}{\csc^2(\frac{\Delta_0}{2}) \sin^2(\xi)}\right). \tag{10}
\]

This integral is performed by classifying into two cases according to the sign of \( \sin(\frac{\xi}{2}) \sin(\frac{\xi}{2}) \equiv \eta \).

[10] In the case of \( \eta > 0 \), we must evaluate the integral (10) for \( \xi \) that satisfies \( 1 - \csc^2(\Delta_0/2) \sin^2(\xi) > 0 \) and \( \xi < \pi/2 \) accordingly to (8) and (10). Therefore, \( \tau \) and \( \xi \) should satisfy the following relations for integers \( m \) and \( n \) \((m = 1, \ldots, n - 1): 2(n-1) \pi + \Delta_0 < \theta < 2n\pi - \Delta_0, 0 < \xi < \Delta_0/2 \) and \( \pi - \Delta_0/2 < \xi < \pi + \Delta_0/2 \). Therefore, the integral region with respect to \( \xi \) is divided into \( n \) discrete parts: \([0, \Delta_0/2), \ldots, \{(n-1)\pi - \Delta_0/2, (n-1)\pi + \Delta_0/2]\)

Namely,

\[
I^{(+)} = \frac{1}{\sin(\Delta_0/2) \sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} \left[ \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}} \right] d\xi + \sum_{n=1}^{n-1} \int_0^{\pi/2} d\xi \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}}
\]

\[
= \frac{2n - 1}{\sin(\Delta_0/2) \sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} \left[ \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}} \right] d\xi + \sum_{n=1}^{n-1} \int_0^{\pi/2} d\xi \frac{1}{\sqrt{1 - \sin^2(\Delta_0/2) \sin^2(\xi)}}
\]

\[
= \frac{2n - 1}{\sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} K \left( \frac{\sin^2(\Delta_0/2)}{2} \right), \tag{11}
\]

where we have introduced integral parameter \( \xi \) as \( \sin \xi \equiv \sin(\Delta_0/2) \sin \zeta \). The superscript \((+\) means the case of \( \eta > 0 \).

The coefficient \( 2n - 1 \) is controlled by the lapse time \( \tau \) and the epicentral distance \( \Delta_0 \), and it corresponds to the number of isochronal scattering curves (see Sato and Nohechi [2001]). Finally, we get

\[
I^{(+)} = \frac{2n - 1}{\sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} K \left( \frac{\sin^2(\Delta_0/2)}{2} \right), \tag{12}
\]

for \( 2(n-1)\pi + \Delta_0 < \theta < 2n\pi - \Delta_0 \),

where the function \( K \) represents a complete elliptic integral of the first kind [Abramowitz and Stegun, 1972, page 590]:

\[
K(m) \equiv \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \nu}} d\nu. \tag{13}
\]

[11] In the case of \( \eta \leq 0 \), similar to the previous case, \( \tau \) and \( \xi \) should satisfy \( 2n\pi - \Delta_0 < \theta < 2n\pi + \Delta_0 \) and \( (m - 1)\pi + \Delta_0/2 < \xi < m\pi - \Delta_0/2 \) for integers \( m \) and \( n \) \((m = 1, \ldots, n \)

Therefore, the integral region is divided into \( n \) parts \([\Delta_0/2, \pi - \Delta_0/2), \ldots, \{(n-1)\pi + \Delta_0/2, \pi - \Delta_0/2]\), and the integral \( I \) is represented as

\[
I^{(-)} = \frac{1}{\sin(\Delta_0/2) \sqrt{\sin^2(\frac{\xi}{2}) \sin^2(\frac{\xi}{2})}} \sum_{n=1}^{n-1} \int_{(n-1)\pi + \Delta_0/2}^{\pi/2} \frac{1}{d\xi}
\]

\[
= -2n \int_{(n-1)\pi + \Delta_0/2}^{\pi/2} \frac{1}{\sin(\Delta_0/2) \sqrt{\sin^2(\frac{\xi}{2}) \sin^2(\frac{\xi}{2})}} \sin(\Delta_0/2) \sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}
\]

\[
= -2n \int_{(n-1)\pi + \Delta_0/2}^{\pi/2} \frac{1}{\sqrt{\sin^2(\Delta_0/2) \sin^2(\xi)}} d\xi. \tag{14}
\]

The superscript \((-\) means the case of \( \eta < 0 \). By introducing integral parameter \( \zeta \) as \( \sin \zeta \equiv \sin(\Delta_0/2) \sin \xi \), we get

\[
I^{(-)} = \frac{2n - 1}{\sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} \left[ \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}} \right] d\xi + \sum_{n=1}^{n-1} \int_{(n-1)\pi + \Delta_0/2}^{\pi/2} \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}} d\xi
\]

\[
= \frac{2n - 1}{\sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} \left[ \frac{1}{\sqrt{1 - \csc^2(\Delta_0/2) \sin^2(\xi)}} \right] d\xi + \sum_{n=1}^{n-1} \int_{(n-1)\pi + \Delta_0/2}^{\pi/2} \frac{1}{\sqrt{1 - \sin^2(\Delta_0/2) \sin^2(\xi)}} d\xi
\]

\[
= \frac{2n - 1}{\sqrt{\sin(\frac{\xi}{2}) \sin(\frac{\xi}{2})}} K \left( \frac{\sin^2(\Delta_0/2)}{2} \right).
\]

where the function \( F \) is an incomplete elliptic integral of the first kind [Abramowitz and Stegun, 1972, page 589):

\[
F(\varphi|m) \equiv \int_0^{\varphi} \frac{1}{\sqrt{1 - m \sin^2 \nu}} d\nu. \tag{16}
\]

As same as the previous case, coefficient \( 2n - 1 \) corresponds to the number of isochronal scattering curves. The integral is performed along the path parallel to the imaginary axis whose real part is \( \pi/2 \), and the term \( K(\sin^2(\Delta_0/2)) - F(\sin^2(\csc(\Delta_0/2))) \) in equation (15) is purely imaginary.

[12] In summary, we get the representation of the single isotropic-scattering energy density for circular radiation as

\[
E^1(\Delta_0, \iota) = \frac{W_{00} \varrho e^{-\varrho_0 \varrho_R}}{4\pi^2 R} \left[ \frac{2n_1(\varrho_1, \varrho_1/R)}{\sqrt{\sin(\varrho_1 + \varrho_1) \sin(\varrho_1 + \varrho_1)}} \right]
\]

\[
= \left\{ \begin{array}{ll}
K \left( \frac{\sin^2(\varrho_1/2)}{2} \right) & \text{for } 2(n-1)\pi + \Delta_0 < \theta < 2n\pi - \Delta_0 \\
K \left( \frac{\sin^2(\varrho_1/2)}{2} \right) - F \left( \sin^{-1} \left[ \csc(\varrho_1/2) \right] \sin^2(\varrho_1/2) \right) & \text{for } 2n\pi - \Delta_0 < \theta < 2\pi + \Delta_0,
\end{array} \right. \tag{17}
\]

where, we introduced the number of isochronal scattering curves as

\[
n_1(\Delta_0, \iota) = \left[ \frac{\tau - \Delta_0}{2\pi} \right] + \left[ \frac{\tau + \Delta_0}{2\pi} \right] + 1. \tag{18}
\]
Figure 2. Temporal change of the essential part of the single scattering energy density \( s(\Delta_0, \tau) \) against normalized lapse time \( \tau \) for two source-receiver distances.

Gauss’s symbol \([x]\) indicates the greatest integer less than or equal to \( x \) (equation (8) in Sato and Nohechi [2001]).

4. Normalized Scattered Energy Density Function

[13] Normalizing the energy density by the total radiated energy and the surface area of the spherical Earth, we get the energy density in non-dimensional form:

\[
e^1(\Delta_0, \tau) = \frac{4\pi R^2}{\mu} E^1(\Delta_0, \tau) = R_0 e^{-g_0} e^{\frac{1}{2} (\Delta_0 - \Delta_0)}.
\]

It is written as a product of the essential part of the single scattering energy density \( s(\Delta_0, \tau) \) and the exponential decay term representing scattering and intrinsic loss.

\[
s(\Delta_0, \tau) = \frac{2n(\Delta_0, \tau)}{\pi \sin \left( \frac{\pi + \Delta_0}{2} \right) \sin \left( \frac{\pi - \Delta_0}{2} \right)},
\]

\[
\begin{align*}
K \left( \frac{\Delta_0}{2} \right) & \quad \text{for } 2(n-1)\pi + \Delta_0 < \tau < 2n\pi - \Delta_0, \\
K \left( \frac{\Delta_0}{2} \right) - \Phi \left( \sin^{-1} \left( \frac{\Delta_0}{2} \right) \right) & \quad \text{for } 2(n-1)\pi - \Delta_0 < \tau < 2n\pi + \Delta_0.
\end{align*}
\]

5. Summary

[14] Based on the single scattering approximation of the radiative transfer theory on the spherical Earth, we have succeeded in deriving an analytic representation of the scattered energy density for the special case of the circular source radiation and isotropic scattering by using an elliptic integral of the first kind. It largely reduces the computer time to synthesize surface wave envelopes compared to the numerical integration.

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References


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