Theoretical background of retrieving Green’s function by cross-correlation: one-dimensional case

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SUMMARY

Recently, an assertion has been verified experimentally and theoretically that Green’s function between two receivers can be reproduced by cross-correlating the records at the receivers. In this paper, we have theoretically proved the assertion for 1-D media with the free surface by using the Thomson–Haskell matrix method. Strictly speaking, one side of the cross-correlation between records at two receivers is the convolution between Green’s function and the autocorrelation function of the source wavelet. This study extends the geometry considered by Claerbout to two receivers vertically apart, and is a special case of the proof by Wapenaar et al. which dealt with 3-D arbitrary inhomogeneous media. However, a simple geometry in 1-D problems enables us to make the proof without any approximations and to better understand the physical background with more ease. That is the main advantage of this study. Though a 1-D geometry seems far from reality, it may be sufficient if an appropriate combination of receivers and earthquakes is selected. In fact, such a geometry is often seen in seismological observations by a vertical array of seismographs in the shallow subsurface. Therefore, we refer to a possibility that the proof in this paper is applied to the estimation of site amplification factors by using records of a vertical seismographic array.

Key words: acoustic daylight imaging, cross-correlation, Green’s function, site amplification factor.

1 INTRODUCTION

Green’s function between two receivers can be retrieved by cross-correlating the records at the receivers. The assertion has been exemplified in ultrasonic laboratory experiments (e.g. Lobkis & Weaver 2001), helioseismology (e.g. Duvall et al. 1993), ocean acoustics (Buckingham et al. 1992) and seismology (Campillo & Paul 2003). Recently, theoretical studies on the assertion are being developed. Lobkis & Weaver (2001) offered a theoretical explanation based on the idea of the equipartition of normal modes which will hold for diffusive waves or later coda waves. Snieder (2004) succeeded in proving the assertion for scalar waves in 3-D purely random media and elastic surface waves in layered media with embedded scatterers. Wapenaar et al. (2004) and Wapenaar (2004) completed the proof for scalar waves and vector waves in 3-D arbitrary inhomogeneous media.

In a field of exploration seismology, the similar idea appeared much earlier. A pioneering paper by Claerbout (1968) showed that one-side of the autocorrelation of a record on the surface for a normal incident wave from below (transmission response) is proportional to the reflection response to a source at the same position on the surface for 1-D cases. According to Rickett (1997), the proof was extended by Claerbout to a conjecture that ‘by cross-correlating noise traces recorded at the surface, we can construct the wavefield that would be recorded at one of the locations if there was an impulsive source at the other’. This conjecture was verified numerically by Rickett & Claerbout (1996) and Rickett (1997). These studies offer a basis for the acoustic daylight imaging (e.g. Rickett & Claerbout 1999), which is a kind of passive technique to image subsurface structures by using ambient noises without any active sources.

In this study, we consider a geometry that two receivers are separated vertically not horizontally in horizontally layered media and prove the Clarebout’s conjecture for the geometry based on the Thomson–Haskell matrix method. The proof for 1-D media in this study is a specific case of the proof by Wapenaar et al. (2004) which dealt with 3-D arbitrary inhomogeneous media in a more general fashion. However, a simple geometry in 1-D problems helps us to make the proof without any approximations and to better understand the physics with more ease. That is the main advantage of this study.

From a practical point of view, such a 1-D geometry is often seen in seismological observations by a vertical array of seismographs in the shallow subsurface. Therefore, we refer to a possibility that the proof in this study is used to the estimation of site amplification factors from records of vertical seismographic arrays.
For the normal incidence of waves, \( \rho \) is denoted as \( \rho L \) where \( w \) is the vertical slowness, which is the reciprocal of the \( k \)-wave velocity \( \beta k \). Here, the following parameters are introduced:

\[ \gamma_k \equiv \frac{\mu_k \eta_k}{\mu_{k+1} \eta_{k+1}} = \frac{\rho_k \beta_k}{\rho_{k+1} \beta_{k+1}} \]

for the normal incidence of waves,

\[ \Delta H_k \equiv H_k - H_k-1. \]

The matrix \( L^{(k+1, k)} \) relates the wave vector of the \((k + 1)\)th layer to that of the \(k\)th layer.

For lossless media, the velocity \( \beta_k \) is real and the following relation holds:

\[ \rho_k \beta_k = \rho_k^* \beta_k^*. \]

Here, the superscript * appearing in the frequency domain stands for the complex conjugation.

For the matrix \( L^{(k, k)} \), the following chain rule holds:

\[ L^{(k, k)} = L^{(k, k-1)} L^{(k-1, k-2)} \ldots L^{(2, 1)}. \]

The following relations are valid among the elements of \( L^{(k, k)} \), which play crucial roles in the subsequent derivation:

\[ L_{11}^{(k, k)} = L_{22}^{(k, k)} \]

\[ L_{12}^{(k, k)} = L_{21}^{(k, k)} \]

2 THEORY

2.1 Thomson–Haskell matrix approach

We consider scalar wave propagation in 1-D lossless media by using the Thomson–Haskell matrix method (e.g. Kennett 1983; Shearer 1999). From a seismological viewpoint, we can imagine a situation that only \( P \) waves or \( S \) waves propagate vertically upwards and downwards in a stack of \( N \) horizontal layers as shown in Fig. 1. We use the notations for \( SH \) waves in the subsequent derivation. However, we note that the resultant relation can be also applied to \( P \) waves only if the normal incidence of waves is considered, because there is no conversion between \( P \) waves and \( S \) waves for the case. Hereafter, we basically follow the notation of Shearer (1999).

Since only the vertical propagation of waves is considered, the horizontal slowness is zero. For the case, the displacement wavefield in \( y \) direction in the \( k \)th layer can be expressed as

\[ u_k = \{ \hat{S}_k \exp[i \omega \eta_k(z - H_{k-1})] + \hat{S}_k \exp[-i \omega \eta_k(z - H_{k-1})] \} \exp(-i \omega t), \]

where \( \hat{S}_k \) and \( \hat{S}_k \) are the amplitudes of a downgoing wave and an upgoing wave, \( \eta_k \) is the vertical slowness, which is the reciprocal of the \( S \)-wave velocity \( \beta_k \), \( H_{k-1} \) is the depth of an interface between the \((k - 1)\)th layer and the \(k\)th layer, and \( \omega \) is the angular frequency. The density is denoted as \( \rho_k \) and the rigidity is \( \mu_k = \rho_k \beta_k^2 \). The derivation is mostly done in the frequency domain. A vector called the wave vector is defined as

\[ w_k \equiv \begin{bmatrix} \hat{S}_k \\ \hat{S}_k \end{bmatrix}. \]

From the boundary condition at an interface at \( z = H_k \) that the displacement and stress across the interface are continuous, we get

\[ w_{k+1} = L^{(k+1, k)} w_k, \]

where

\[ L^{(k+1, k)} = \frac{1}{2} \begin{bmatrix} (1 + \gamma_k) \exp(i \omega \eta_k \Delta H_k) (1 - \gamma_k) \exp(-i \omega \eta_k \Delta H_k) \\ (1 - \gamma_k) \exp(i \omega \eta_k \Delta H_k) (1 + \gamma_k) \exp(-i \omega \eta_k \Delta H_k) \end{bmatrix}. \]
2.2 Response to normal incidence of waves from below

First, we consider a normal incidence of waves from below. Such a geometry is often considered for the estimation of the site (near-surface structure) response to normal incident waves from below.

For wave vectors,

\[ w_N = L^{(N,1)}w_1. \]  

(11)

From the boundary condition at the free surface,

\[ \hat{S}_1 = \tilde{S}_1 = 0. \]  

(12)

By using the relations (11) and (12), the displacement at the surface is expressed as

\[ u_0 = 2S_1 = \frac{2 S_N}{L_{21}^{(N,1)} + L_{22}^{(N,1)}}. \]  

(13)

Though \( u_0 \) should be written strictly as \( u_0(\omega; t) \), the abbreviated form \( u_0 \) is used hereafter.

For a subsurface receiver in the \( k \)th layer, we calculate the displacement at the receiver. The wave vector in the \( k \)th layer satisfies the similar relation to (11). Therefore,

\[ u_k = \hat{S}_k \exp(i \omega \Delta H_k) + \hat{S}_k \exp(-i \omega \Delta H_k) \]

\[ = \frac{S'_N}{L_{21}^{(N,1)} + L_{22}^{(N,1)}} \left[ (L_{11}^{(1,1)} + L_{12}^{(1,1)}) \exp(i \omega \Delta H_k) + (L_{21}^{(1,1)} + L_{22}^{(1,1)}) \exp(-i \omega \Delta H_k) \right]. \]

(14)

where

\[ \Delta H_k \equiv z_s - H_{k-1}, \]  

(15)

and \( z_s \) is the depth of the subsurface receiver.

The cross-correlation between (13) and (14) is as follows:

\[ u_0 u_k^* \]

\[ = \frac{2|S_N|^2}{|L_{21}^{(N,1)} + L_{22}^{(N,1)}|^2} \left[ (L_{11}^{(1,1)} + L_{12}^{(1,1)}) \exp(-i \omega \Delta H_k) + (L_{21}^{(1,1)} + L_{22}^{(1,1)}) \exp(i \omega \Delta H_k) \right]. \]

(16)

2.3 Green’s function at a subsurface receiver for a source at the surface

Here, we consider Green’s function at the subsurface receiver for a source at the surface. A source can be treated as the discontinuity of the wave vector (e.g. Kennett 1983). The discontinuity of a wave vector \( w_s \) at the source depth \( z_s \) is related to the discontinuity of both displacement \( d_s \) and stress \( d_s^* \) at the same depth as

\[ w_s = \frac{1}{2} \begin{bmatrix} d_s & d_s^* \end{bmatrix} \left( \begin{array}{c} d_d + d_s^* \omega \eta k \beta \Delta t_1 \\ d_d - d_s^* \omega \eta k \beta \Delta t_1 \end{array} \right). \]

(17)

Green’s function is an impulse response to a unit-amplitude body force. The discontinuity of stress is equivalent to a body force. Hence, we consider an impulsive source with a unit-amplitude discontinuity of stress at the surface. We note that the discontinuity of displacement is not realizable at the free surface. For the case, the discontinuity of the wave vector at the source is

\[ w_s^{(SS)} = \frac{1}{2 \omega \eta k \beta} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

(18)

The superscript (SS) means the surface source.

The wave vector in the bottom half-space is

\[ w_N = L^{(N,1)}[w_1 + w_s^{(SS)}]. \]

(19)

From the radiation condition

\[ \hat{S}_N = 0, \]  

(20)

we obtain

\[ S_1 = \frac{w_s^{(SS)} (L_{22}^{(N,1)} - L_{21}^{(N,1)})}{L_{21}^{(N,1)} + L_{22}^{(N,1)}}. \]

(21)
Using a relation similar to (19) for \( \mathbf{w}_s \), we obtain
\[
\dot{\mathbf{S}}_a = 2 w_{SS} (S^{(N,1)}_{11} - L_{21}^{(N,1)} L_{12}^{(N,1)}) \frac{L_{22}^{(N,1)} - L_{21}^{(N,1)} L_{12}^{(N,1)}}{L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)}}.
\] (22)
and
\[
\dot{\mathbf{S}}_a = 2 w_{SS} (S^{(N,1)}_{22} - L_{22}^{(N,1)} L_{12}^{(N,1)}) \frac{L_{22}^{(N,1)} - L_{21}^{(N,1)} L_{12}^{(N,1)}}{L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)}}.
\] (23)

Therefore, the displacement at the subsurface receiver due to a surface source denoted as \( u_d^{SSG} \), where the superscript ‘G’ means Green’s function, is
\[
u_d^{SSG} = 2 w_{SS} \left[ (S^{(N,1)}_{11} - L_{21}^{(N,1)} L_{12}^{(N,1)}) \frac{L_{22}^{(N,1)} - L_{21}^{(N,1)} L_{12}^{(N,1)}}{L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)}} \right] \exp(i \omega \eta_1 \Delta H_2) \left[ (S^{(N,1)}_{22} - L_{22}^{(N,1)} L_{12}^{(N,1)}) \frac{L_{22}^{(N,1)} - L_{21}^{(N,1)} L_{12}^{(N,1)}}{L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)}} \right] \exp(-i \omega \eta_1 \Delta H_2).
\] (24)

Because Green’s function satisfies the causality, time-domain Green’s function is zero for time \( t < 0 \).

### 2.4 Relation between cross-correlation and Green’s function

Multiplying both numerator and denominator of equation (24) with \( [L_{21}^{(N,1)} + L_{22}^{(N,1)}] \), multiplying those of the complex conjugates (or a time-reversed trace) of Green’s function \( u_d^{SSG} \), and using the relations (9) and (10), we obtain the following relation:
\[
u_d^{SSG} = (u_d^{SSG})^* = 2 w_{SS} \left[ L_{21}^{(N,1)} \left( L_{11}^{(N,1)} + L_{12}^{(N,1)} \right)^2 \right] \left( L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)} \right) \exp(i \omega \eta_1 \Delta H_2) \left( L_{22}^{(N,1)} + L_{21}^{(N,1)} L_{12}^{(N,1)} \right) \exp(-i \omega \eta_1 \Delta H_2).
\] (25)

Comparing the above equation with (16) and using (A2) in Appendix A, we obtain
\[\nu_0 u_0 \equiv 2 \omega_0 \beta_N \tilde{\mathbf{S}}_N \tilde{\mathbf{S}}_N^* \left( u_d^{SSG} - (u_d^{SSG})^* \right). \] (26)

Multiplying both sides of (26) by \((-i \omega_0)\), we derive an important relation:
\[(-i \omega_0) \nu_0 u_0 \equiv 2 \omega_0 \beta_N \beta_0 \tilde{\mathbf{S}}_N \tilde{\mathbf{S}}_N^* \left[ u_d^{SSG} - (u_d^{SSG})^* \right]. \] (27)

So far, the derivation is done in the frequency domain. The relation is expressed in the time domain as
\[
\frac{d}{dt} \int \nu_0(t) u_d(t + \tau) d\tau = 2 \omega_0 \beta_N R_{\mathcal{S}'_N(t)}(t) \ast \left[ u_d^{SSG}(t) - u_d^{SSG}(-t) \right],
\] (28)
where
\[
R_{\mathcal{S}'_N(t)}(t) = \int \tilde{\mathcal{S}}'_N(t) \tilde{\mathcal{S}}'_N(t + \tau) d\tau,
\] (29)
and \( \mathcal{S}_N(t) \) stands for a time trace of an input upgoing wave in the bottom half-space. We call the time derivative of \( \mathcal{S}_N(t) \), that is, \( \mathcal{S}_N'(t) \), as the source wavelet in this paper. Symbols \( \ast \) and \( \tau \) appearing in the time domain mean the convolution and the time derivative, respectively. The necessity of the time differentiation of the cross-correlation was pointed out by Sabra et al. (2005).

For velocity records \( v = (-i \omega_0) u \), we can derive similar relations to (27) and (28). Multiplying both sides of (27) by \( i \omega_0 \), we find
\[
\nu_0 v_0 = -2 \omega_0 \beta_N \beta_0 \tilde{\mathbf{S}}_N \left[ v_d^{SSG} + (v_d^{SSG})^* \right].
\] (30)

In the time domain,
\[
\int \nu_0(t) v_d(t + \tau) d\tau = -2 \omega_0 \beta_N R_{\mathcal{S}'_N(t)}(t) \ast \left[ v_d^{SSG}(t) + v_d^{SSG}(-t) \right].
\] (31)

We have thus completed the proof that the cross-correlation between records at two receivers is the convolution between Green’s function and the autocorrelation function of the source wavelet. As long as the autocorrelation function of the source wavelet is close to the delta function, Green’s function can be retrieved from cross-correlation of observed records.

### 3 Numerical Experiment

Here, we make some numerical experiments and try to confirm the relation (31). For that purpose, I adopt a model with two layers on the half-space shown in Fig. 2. The model is set by referring to Daneshvar et al. (1995). A subsurface receiver is located in the second layer. We can use theoretical expressions derived in the previous section for the layer number \( N \) of 3 and the source layer number \( k \) of 2 in this case.

For the numerical simulation, the fast Fourier transform (FFT) is used to transform from the frequency domain to the time domain. The sampling interval is 0.01 s and the length of records is 20.48 s. Practically, amplitudes of traces become almost zero after 5 s in our simulations. In this geometry, a plane wave with unit amplitude is vertically incident on the layered structure from below. For simplicity, the source wavelet, \( S_N(t) \), is assumed to be the delta function (practically a box-car function with the duration of one sampling interval). Velocity records at the surface and the subsurface are shown in Fig. 3. Here, the unit is chosen in such a way that the incident wave has a unit amplitude.

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When the incident wave touches the interface between the second and the third layer (the second interface), the time is set to be zero. For the record at the subsurface (Fig. 3a), the first pulse arrives at 0.02 s. The second pulse is a reflected phase at the interface between the first and the second layer (the first interface), and the third pulse is the reflected phase of the second pulse at the second interface. In this way, many later phases can be generated. For the record at the surface (Fig. 3b), the first pulse is the direct phase from below and appears at the time of 0.32 s. The second pulse is the reflected phase of the first pulse at the first interface. Similarly, many later phases appear. It is needless to say that these two traces are causal. From the two records, the cross-correlation trace for time $t < 0$ and that for $t > 0$ are shown in Figs 3(c) and (d). These traces are symmetrical with respect to the time reversal $t \rightarrow -t$, in agreement with (38). On the trace (d), the first pulse appears at the time of 0.3 s which is the traveltime between the two receivers.

As the second step, we calculate Green’s function for velocity record at the subsurface receiver for a source at the surface with a unit-amplitude stress discontinuity. For this case, Green’s functions at the surface and the subsurface are shown in Fig. 4.

As the third step, we calculate Green’s function for velocity at the surface receiver for a source at the subsurface with a unit-amplitude stress discontinuity. For this case, Green’s functions at the surface and the subsurface are shown in Fig. 5.

Finally in Fig. 6, we compare the cross-correlation of velocity traces at the two receivers for normal incident waves, Green’s functions for velocity at the subsurface for the source at the surface, and that at the surface for the source at the subsurface. All traces are normalized for comparison, and are found to have the same shape. So, Green’s function between two receivers can in fact be extracted from the cross-correlation of two traces at the surface and the subsurface. As a more quantitative argument, the cross-correlation is expected to be $2\rho_3\beta_3 = 20$ times Green’s function at the subsurface for a source at the surface from (31). This is confirmed in Fig. 7.
Figure 4. Response to a source at the surface. Velocity record with the polarity reversed, \( -v^{\text{SSG}}(t) \), at the subsurface receiver (top panel), and that at the surface receiver (bottom panel).

Figure 5. Response to a source at the subsurface. Velocity record with the polarity reversed, \( -v^{\text{DSG}}(t) \), at the subsurface receiver (top panel), and that at the surface receiver (bottom panel).

4 DISCUSSION

4.1 Reciprocal relation

Here, we describe the reciprocal relation between Green’s function at the surface for a subsurface source and that at the subsurface for a surface source.

First, a subsurface source at \( z = z_s \) in the \( k \)th layer, denoted as (DS) meaning the deep source, is assumed to have a unit-amplitude stress discontinuity. For the case, the discontinuity of the wave vector at \( z = z_s \) is written as

\[
w^{\text{(DS)}}_s = \left[ \frac{1}{2i\omega\rho\beta_k} + i \frac{1}{2i\omega\rho\beta_k} \right]\]

\[\times \left[ \begin{array}{c} u^{\text{(DS)}}_s \\ -u^{\text{(DS)}}_s \end{array} \right]. \tag{32}\]

From the boundary conditions at \( z = H_{k-1} \), we obtain

\[
w_N = L^{(N,1)}w_1 + L^{(N,1)} \left[ \begin{array}{c} u^{\text{(DS)}}_s \exp(-i\omega\eta_1 \Delta H_s) \\ -u^{\text{(DS)}}_s \exp(i\omega\eta_1 \Delta H_s) \end{array} \right]. \tag{33}\]

From (33) and the radiation condition of (20), displacement at the surface for the subsurface source is written as

\[
u_0^{\text{DSG}} = 2S_1 = \frac{-2u^{\text{(DS)}}_s L^{(N,1)}_{21}}{L^{(N,1)}_{21} + L^{(N,1)}_{22}} \exp(-i\omega\eta_1 \Delta H_s) + \frac{2u^{\text{(DS)}}_s L^{(N,1)}_{21}}{L^{(N,1)}_{21} + L^{(N,1)}_{22}} \exp(i\omega\eta_1 \Delta H_s). \tag{34}\]

From the chain rule of (8),

\[
L^{(N,1)} = L^{(N,1)}(L^{(k,1)})^{-1}. \tag{35}\]
Green’s functions from cross-correlation

Figure 6. Comparison among the cross-correlation between the surface record and subsurface record for normal incident waves (top panel), the polarity-reversed velocity record at the subsurface receiver for the surface source (middle panel), the polarity-reversed velocity record at the surface receiver for the subsurface source (bottom panel). All traces are normalized only in this figure.

Figure 7. Quantitative comparison between the cross-correlation between the surface record and subsurface record for normal incident waves (top panel) and the polarity-reversed velocity record at the subsurface receiver for the surface source (bottom panel). The difference in amplitude by 20 times between the records is exactly the same as expected from the theory.

Here, the superscript $-1$ means the inverse matrix and

$$(L^{(k,1)})^{-1} = \frac{1}{\gamma_k - 1} \begin{bmatrix} L_{22}^{(k,1)} & -L_{12}^{(k,1)} \\ -L_{21}^{(k,1)} & L_{11}^{(k,1)} \end{bmatrix}. \quad (36)$$

Applying (36) recursively, we obtain

$$(L^{(k,1)})^{-1} = (L^{(2,1)})^{-1}(L^{(3,2)})^{-1} \cdots (L^{(k,k-1)})^{-1}$$

$$= \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_{k-1}} \begin{bmatrix} L_{22}^{(1)} & -L_{12}^{(1)} \\ -L_{21}^{(1)} & L_{11}^{(1)} \end{bmatrix}. \quad (37)$$

Using (32), (34), (35) and (37), we obtain

$$u_0^{(DS)G} = \frac{2w^{(DS)}}{\gamma_1 \gamma_2 \cdots \gamma_{k-1}} \left[ L_{22}^{(N,1)} L_{11}^{(k,1)} - L_{21}^{(N,1)} L_{12}^{(k,1)} \right] \exp(i\omega \eta_k \Delta H_s)$$

$$+ \left[ L_{22}^{(N,1)} L_{21}^{(k,1)} - L_{21}^{(N,1)} L_{22}^{(k,1)} \right] \exp(-i\omega \eta_k \Delta H_s). \quad (38)$$

Comparing (38) with (24), we obtain

$$u_0^{(DS)G} = \eta_0^{(SS)G}. \quad (39)$$

The reciprocity of Green’s function is proved in this way.

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4.2 Symmetry of cross-correlation with time

As long as the relations (9) and (10) hold, it is proved by taking the complex conjugation of (16) that

\[ u_0^\ast u_d = u_0 u_d^\ast. \]  

(40)

Then, the cross-correlation is symmetrical with time.

4.3 Reproducing the proof by Claerbout (1968)

Claerbout (1968) proved that the reflection response can be obtained from the autocorrelation of the transmission response at the surface. The proof was done by using the Z transform. Here, we try to reproduce the proof by Claerbout following our notation.

Autocorrelation of the transmission response, which is displacement at the surface for a normal incident wave from below, is

\[ u_0 u_0^\ast = \frac{4|S_N|^2}{L_{21}^{N,N} + L_{22}^{N,N}}. \]  

(41)

For a source with a unit-amplitude stress discontinuity at the surface, Green's function at the same location can be expressed as

\[ \varphi_0^{(SS)} = \frac{2u_s^{SS} L_{21}^{N,N} - L_{22}^{N,N}}{L_{21}^{N,N} + L_{22}^{N,N}}. \]  

(42)

Using \( \varphi_0^{(SS)} \), its complex conjugate \( (\varphi_0^{(SS)})^\ast \), and eq. A2, we obtain

\[ u_0 u_0^\ast = \frac{|S_N|^2}{u_s^{SS}} \frac{\rho_N \beta_N}{\rho_1 \beta_1} \left[ u_0^{SSG} - (u_0^{SSG})^\ast \right]. \]  

(43)

In Claerbout (1968), the reflection response is calculated for a source which impulsively radiates only a unit-amplitude downgoing wave just below the free surface. Though such a source is not physically realizable as shown in (17), the reflection response, denoted as \( R^{SS} \), can be expressed theoretically as

\[ R^{SS} = \frac{-2L_{21}^{N,N}}{L_{21}^{N,N} + L_{22}^{N,N}}. \]  

(44)

and the following relation holds:

\[ u_0^{SSG} = 2u_s^{SY} \left[ 1 + R^{SS} \right]. \]  

(45)

Green's function is composed of the source signal itself and the reflection response.

Substituting (45) into (43), the following relation is obtained:

\[ u_0 u_0^\ast = 2|S_N|^2 \frac{\rho_N \beta_N}{\rho_1 \beta_1} \left[ R^{SS} + (R^{SS})^\ast + 2 \right]. \]  

(46)

In the time domain, the relation is

\[ \int u_0(t) u_0(t + \tau) d\tau = 2 \frac{\rho_N \beta_N}{\rho_1 \beta_1} R_{SS}(t) \ast \left[ R^{SS}(t) + R^{SS}(-t) + 2\delta(t) \right]. \]  

(47)

A corresponding expression to Claerbout (1968) is thus reproduced, though the coefficient of the delta function is changed to 2 due to the amplification at the free surface.

4.4 Effect of attenuation

So far, lossless media are considered. What is the effect of the attenuation? As long as the attenuation of waves is small, the effect can be incorporated by introducing an imaginary part in wave velocity. Taking the reference frequency as 1 Hz, the complex wave velocity is expressed as

\[ \beta(\omega) \approx \beta(\omega = 2\pi) \left[ 1 + \frac{1}{\pi Q} \ln \left( \frac{\omega}{2\pi} \right) - i \text{sgn}(\omega) \frac{1}{2Q} \right] \]  

(Kennett 1983).

When the wave velocity \( \beta \) becomes complex, the relations of (9) and (10) for the matrix \( L^{(k+1,k)} \) no longer hold. Therefore, the time symmetry of the cross-correlation does not hold for dissipative media. The value of the cross-correlation in negative time lags is larger than that in positive time lags. In addition, Green's function may not be retrieved by cross-correlation exactly. However, as long as the imaginary part of the velocity is small, the relations of (9) and (10) for the matrix \( L^{(k+1,k)} \) may approximately hold. That may be a reason why Green's function is reproduced through the practical data analysis.

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4.5 Implications for practical use

Here, we discuss a possibility for a practical use of the proof in this study. As shown above, a 1-D problem helps us to well understand the physical background of why Green’s function can be retrieved from cross-correlation. A 1-D geometry may seem far from reality. However, the successful application of Claerbout’s (1968) proof to practical data by Scherbaum (1987b) and Daneshvar et al. (1995) showed that 1-D treatment is sufficient as long as appropriate sets of events and receivers are selected.

Now we speculate that the proof in this study can be applied to the estimation of the site response (Green’s function between the surface and the subsurface receivers) by using a vertical array of seismographs (e.g. Izumi et al. 1989). However, the issue whether the autocorrelation function of the source wavelet is the delta function or not is important, because the cross-correlation between records at two receivers is the convolution between Green’s function and the autocorrelation function of the source wavelet. For practical use, further consideration on reducing the effect of the autocorrelation function of the source wavelet should be needed. An interesting and important question is to what degree the random (or stochastic) heterogeneity in the lower half-space makes the source wavelet white-like and deteriorates the convolution of the source wavelet. This point was discussed in Daneshvar et al. (1995) and Rickett (1997). In addition, the blind deconvolution technique which Scherbaum (1987a) made use of is another practical solution for the purpose.

5 CONCLUSION

Using the Thomson–Haskell matrix method, we have proved that Green’s function between two receivers can be extracted by cross-correlating the records at the receivers for 1-D cases. Strictly speaking, the restriction of the cross-correlation function to positive time lags is the convolution between Green’s function and the autocorrelation function of the source wavelet. The geometry considered by Claerbout (1968) was extended to two receivers vertically apart, in this study. The proof in this study is a special case of the proof by Wapenaar et al. (2004). Though 1-D geometry seems far from reality, the simple geometry enables us to make a strict proof and to better understand the physics with more ease. And even 1-D geometry is sufficient if an appropriate combination of receivers and earthquakes is selected. From a practical viewpoint, the proof in this paper can be applied to the estimation of site amplification factors from records of vertical seismographic arrays after a few practical difficulties are solved.

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\[ |L_{22}^{(N,1)}|^2 - |L_{21}^{(N,1)}|^2 \]
\[ = \left( \frac{1}{2} \right)^2 \left| (1 - \gamma_{N-1}) \exp(i\omega\eta_{N-1} \Delta H_{N-1}) L_{12}^{(N-1,1)} + (1 + \gamma_{N-1}) \exp(-i\omega\eta_{N-1} \Delta H_{N-1}) L_{22}^{(N-1,1)} \right|^2 \]
\[ - \left( \frac{1}{2} \right)^2 \left| (1 - \gamma_{N-1}) \exp(i\omega\eta_{N-1} \Delta H_{N-1}) L_{11}^{(N-1,1)} + (1 + \gamma_{N-1}) \exp(-i\omega\eta_{N-1} \Delta H_{N-1}) L_{21}^{(N-1,1)} \right|^2 \]
\[ = \left( \frac{1}{2} \right)^2 \left( |L_{22}^{(N-1,1)}|^2 - |L_{21}^{(N-1,1)}|^2 \right) \left( |(1 + \gamma_{N-1})^2 - |(1 - \gamma_{N-1})|^2 | \right) \]
\[ = (|L_{22}^{(N-1,1)}|^2 - |L_{21}^{(N-1,1)}|^2) \gamma_{N-1}. \] (49)

Applying this relation to \( |L_{22}^{(N-1,1)}|^2 - |L_{21}^{(N-1,1)}|^2 \) recursively, we obtain

\[ |L_{22}^{(N,1)}|^2 - |L_{21}^{(N,1)}|^2 = \gamma_1 \gamma_2 \cdots \gamma_{N-1} = \frac{\beta_1 \beta_2}{\beta_{N-1} \beta_N}. \] (50)