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# Formulation of the spatial autocorrelation (SPAC) method in dissipative media

Hisashi Nakahara

*Department of Geophysics, Graduate School of Science, Tohoku University, Aoba-ku, Sendai* 980–8578*, Japan. E-mail: naka@zisin.gp.tohoku.ac.jp*

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## SUMMARY

Spatial autocorrelation (SPAC) method is formulated in dissipative media for one-, two- and three-dimensional (1-D, 2-D and 3-D) scalar wavefields based on the generalized wave equation or the generalized telegraph equation. A rather straightforward derivation is possible by using a close mathematical relation between the SPAC method and seismic interferometry, though a model set-up should be modified by including attenuation. For 3-D cases, the normalized cross spectrum of a scalar wavefield in a dissipative medium is found to be different from that in a non-dissipative medium merely by an exponentially damping term. However, expressions for 1-D and 2-D cases are not as simple as 3-D cases meaning that not only amplitude but also phase in the normalized cross-spectrum are modified by attenuation. The SPAC expressions derived are considered to be applied to rather homogeneous distribution of attenuation. For the estimation of attenuation, the SPAC method needs to be used for larger station separations than the traditional SPAC method does. Analysis of the SPAC expressions for 2-D cases shows that the conjecture of Prieto *et al.* (2009) is not strict but approximately good for small attenuation. This study will provide a theoretical basis to estimate not only phase velocity but also attenuation from analysis of ambient noises.

**Key words:** Seismic attenuation; Wave propagation.



## 1 INTRODUCTION

Spatial autocorrelation (SPAC) method was proposed by Aki (1957) to estimate surface-wave phase velocity from correlation of ambient noises. The SPAC method has been applied to smaller station separations (e.g. Okada 2003). Seismic interferometry or the noise correlation method (e.g. Campillo & Paul 2003; Curtis *et al.* 2006) has been extensively used to estimate velocity structures and its temporal variations (e.g. Shapiro *et al.* 2005; Sens-Schonfelder & Wegler 2006). Usually seismic interferometry is used for station separations much larger than a concerned wavelength. In this range, applicability of seismic interferometry is enhanced by a stationary phase approximation (e.g. Snieder 2004). Because the SPAC method is shown to be akin to seismic interferometry by theoretical studies (e.g. Nakahara 2006; Yokoi & Margaryan 2008), knowledge on seismic interferometry can be applied to the SPAC method as well. Seismic interferometry is proved to be valid for any type of heterogeneous structures (e.g. Wapenaar & Fokkema 2006). This suggests that the SPAC method is applied to heterogeneous velocity structures. The SPAC method originally derived for media without dissipation, but this is not serious for smaller station separations where the effect of dissipation may be small.

Attenuation was practically measured for a building using deconvolution interferometry (e.g. Snieder & Safak 2006). Later, Prieto *et al.* (2009) conjectured a SPAC expression for surface waves, and estimated an attenuation structure in California. However, a fact that attenuation can be determined from deconvolution interferometry as well as the SPAC method has not been clear theoretically. Recently, seismic interferometry was proved theoretically for scalar waves in dissipative media (e.g. Snieder 2007; Weaver 2008; Margerin & Sato 2011). Numerical approaches were also taken to show the validity as Cupillard & Capdeville (2010) and Weaver (2011). Concerning the SPAC method, Tsai (2011) made a theoretical consideration on the expression of Prieto *et al.* (2009). Though he started his derivation from a damped wave equation, he implicitly dealt with a different equation by imposing a small attenuation. So the derivation does not seem consistent from a theoretical view point.

In this study, I derive SPAC expressions in a straightforward and consistent way for 1-D, 2-D and 3-D cases in homogeneous dissipative media by the help of the recent developments in seismic interferometry for dissipative media.

## 2 FORMULATION

### 2.1 Relations between the SPAC method and seismic interferometry

First, I summarize a theoretical relation between the SPAC method and seismic interferometry based on previous studies for

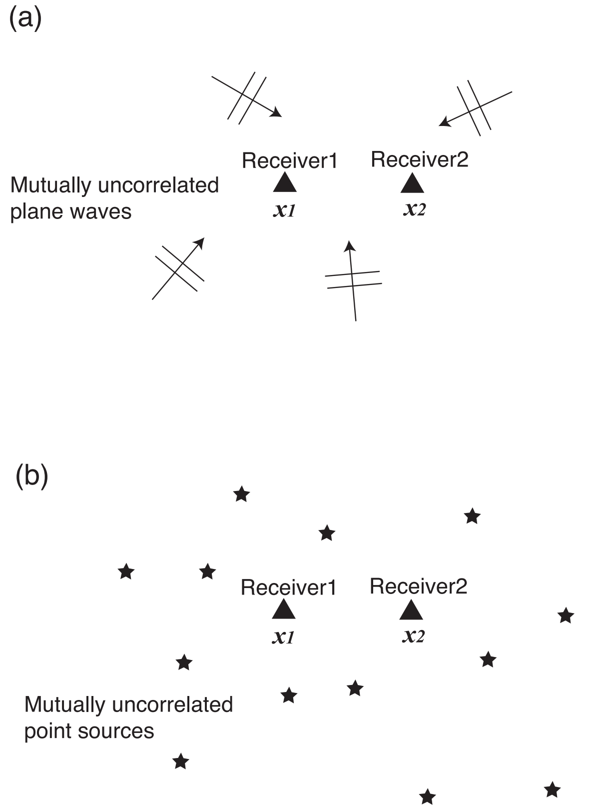



**Figure 1.** (a) Set-up of a model for a non-dissipative medium in which mutually uncorrelated plain waves are incident on two receivers from various directions. (b) Set-up of a model for a dissipative medium throughout which noise sources are distributed homogeneously and randomly.

non-attenuating media. According to Aki (1957), a normalized cross spectrum $C_{1,2}(\mathbf{r}, \omega)$ of a random scalar wavefield at two receivers at $\mathbf{x}_1$ and $\mathbf{x}_2$ is defined as

$$C_{1,2}(\mathbf{r}, \omega) \equiv \frac{< u(\mathbf{x}_1, \omega) u(\mathbf{x}_2, \omega)^* >}{< |u(\mathbf{x}_1, \omega)|^2 >} = \frac{< u(\mathbf{x}_1, \omega) u(\mathbf{x}_2, \omega)^* >}{< |u(\mathbf{x}_2, \omega)|^2 >}, \tag{1}$$

where $< >$ means an ensemble average, $*$ denotes complex conjugate, $u$ is the scalar wavefield, $\omega$ is angular frequency, $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ and $r = |\mathbf{r}|$. For the last equality, the spatial stationarity of the field as $< |u(\mathbf{x}_1, \omega)|^2 > = < |u(\mathbf{x}_2, \omega)|^2 >$ is assumed. An original set-up for non-dissipative cases is as shown in Fig. 1(a) in which mutually uncorrelated plane waves are considered to be incident from various directions.

According to theoretical developments in seismic interferometry (e.g. Lobkis & Weaver 2001; Wapenaar 2004; Sanchez-Sesma & Campillo 2006), Green's functions between two receivers can be retrieved from cross correlation between waves at the two receivers in a medium without dissipation. According to Sanchez-Sesma & Campillo (2006), the following relations hold:

$$< u(\mathbf{x}_1, \omega) u(\mathbf{x}_2, \omega)^* > \propto \mathrm{Im}[G(\mathbf{x}_1, \mathbf{x}_2, \omega)], \tag{2}$$

$$< |u(\mathbf{x}_1, \omega)|^2 > \propto \mathrm{Im}[G(\mathbf{x}_1, \mathbf{x}_1, \omega)], \tag{3}$$

where Im means imaginary part, and the ensemble average in seismic interferometry corresponds mathematically to integration with respect to solid angles of incident waves. Substituting the two equations into eq. (1), the normalized cross spectrum can be described:

$$\begin{aligned} C_{1,2}(\mathbf{r}, \omega) &= \frac{< u(\mathbf{x}_1, \omega) u(\mathbf{x}_2, \omega)^* >}{< |u(\mathbf{x}_1, \omega)|^2 >} \\ &= \frac{\mathrm{Im}[G(\mathbf{x}_1, \mathbf{x}_2, \omega)]}{\mathrm{Im}[G(\mathbf{x}_1, \mathbf{x}_1, \omega)]} \\ &= \frac{\mathrm{Im}[G(\mathbf{r}, \omega)]}{\lim_{\mathbf{r}\to 0} \mathrm{Im}[G(\mathbf{r}, \omega)]}. \end{aligned} \tag{4}$$

This is the theoretical relation connecting the SPAC method to seismic interferometry, which plays a central role in this study. Similar relation was shown for loss-less elastic-wave cases by Yokoi & Margaryan (2008). Here we point out that eq. (4) holds as long as seismic interferometry is valid regardless whether a medium is dissipative or not. This point is mentioned again in Section 2.3.

### 2.2 Reproducing SPAC expressions in non-dissipative media

Here, I reproduce derivation of SPAC expressions for non-dissipative media, because this is logically parallel with derivation for dissipative media shown later. Before starting the derivation, I make a comment on the SPAC method for non-dissipative media. The SPAC method holds even for a non-isotropic wavefield, because the normalized cross-spectrum can be averaged with respect to various incident directions by using the wavefield at the centre of a circle and the wavefields on a circumference of the circle. On the other hand, seismic interferometry holds for an isotropic wavefield in a strict sense. Therefore, I deal with the isotropic wavefield in the following. But a non-isotropic wavefield can be considered like Nakahara (2006).

Green's functions for wave equation in infinite homogenous media are as follows (e.g. Snieder 2001):

$$G(\mathbf{r}, \omega) = -\frac{i \exp(i k_0 r)}{2k_0} \quad \text{for} \quad \text{1-D}, \tag{5}$$

where $k_0 = \omega / V_0$ stands for wavenumber with constant velocity $V_0$,

$$G(\mathbf{r}, \omega) = -\frac{i}{4} H_0^{(1)}(k_0 r) \quad \text{for} \quad \text{2-D}, \tag{6}$$

where $H_0^{(1)}$ means the zeroth order Hankel function of the first kind, and

$$G(\mathbf{r}, \omega) = -\frac{\exp(i k_0 r)}{4\pi r} \quad \text{for} \quad \text{3-D}. \tag{7}$$

Therefore, imaginary parts of the Green's functions are

$$\mathrm{Im}[G(\mathbf{r}, \omega)] = \frac{-\cos(k_0 r)}{2k} \quad \text{for} \quad \text{1-D}, \tag{8}$$

$$\mathrm{Im}[G(\mathbf{r}, \omega)] = \frac{-J_0(k_0 r)}{4} \quad \text{for} \quad \text{2-D}, \tag{9}$$

where $J_0$ means the zeroth order Bessel function of the first kind, and

$$\mathrm{Im}[G(\mathbf{r}, \omega)] = \frac{-\sin(k_0 r)}{4\pi r} \quad \text{for} \quad \text{3-D}. \tag{10}$$

Substituting them into eq. (4) and taking care of the limit of $r \to 0$ in the denominator, the SPAC expressions for 1-D, 2-D and 3-D random scalar wavefields can be obtained as:

$$C_{1,2}(\mathbf{r}, \omega) = \cos(k_0 r) \quad \text{for} \quad \text{1-D}, \tag{11}$$

$$C_{1,2}(\mathbf{r},\omega) = J_0(k_0 r) \quad \text{for} \quad \text{2-D}, \tag{12}$$

$$C_{1,2}(\mathbf{r},\omega) = \frac{\sin(k_0 r)}{k_0 r} \quad \text{for} \quad \text{3-D}. \tag{13}$$

These results are consistent with previous studies (e.g. Aki 1957; Nakahara 2006).

### 2.3 Deriving SPAC expressions in dissipative media

Derivation of SPAC expressions for attenuating media is parallel with one for non-attenuating media. But there is a big difference in a set-up of a model. Seismic interferometry is proved to be valid even for attenuating media (e.g. Snieder 2007; Weaver 2008; Margerin & Sato 2011). Once seismic interferometry holds, eq. (4) holds even for dissipative media. But Snieder (2007) clarified that noise sources must exist throughout a medium volumetrically so that generated energy from the sources balances with dissipation. So the model set-up considered for dissipative media should be as shown in Fig. 1(b), which contrasts with a non-dissipative case shown in Fig. 1(a).

Margerin & Sato (2011) proved the validity of seismic interferometry for scalar waves in a medium having spatially heterogeneous velocities but a homogeneous attenuation coefficient. The wave equation treated is:

$$\Delta u(\mathbf{r},t) - \left( \frac{1}{V_0^2}\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2} + \frac{2\kappa}{V_0}\frac{\partial u(\mathbf{r},t)}{\partial t} + \kappa^2 u(\mathbf{r},t) \right) = 0. \tag{14}$$

Here, $u$ is the wavefield, $\Delta$ is the Laplacian operator and $\kappa$ stands for attenuation. This equation is called generalized wave equation by Imamura (1978) or generalized telegraph equation by Courant & Hilbert (1962). In this equation, the attenuation term is proportional to the time derivative of $u$. For homogeneous distribution of velocity and attenuation, seismic interferometry can be proved directly by a use of the resolvent formula as shown in Sato *et al.* (2012). The ensemble averages in eq. (4) mathematically correspond to a volume integral with respect to noise source distributions for dissipative media. The Green's functions in space and time domain for eq. (14) in infinite homogeneous media are found in Imamura (1978), and are easily transformed into frequency domain. Here, I derive the SPAC expressions in the order of 1-D, 3-D and 2-D, because some longer manipulations are necessary for 2-D case.

The 1-D case is straightforward. Green's function is

$$G(\mathbf{r},\omega) = -\frac{i\exp\left(i(k_0+i\kappa)r\right)}{2(k_0+i\kappa)}. \tag{15}$$

Comparison with eq. (5) shows that attenuation can be included by using a complex wavenumber. Imaginary part of the Green's function is

$$\mathrm{Im}[G(\mathbf{r},\omega)] = -\frac{\exp(-\kappa r)}{2k_0}\frac{\cos k_0 r + (\kappa/k_0)\sin k_0 r}{1+(\kappa/k_0)^2}. \tag{16}$$

Then substituting the equation into eq. (4), the following relation is obtained:

$$C_{1,2}(\mathbf{r},\omega) = \left[\cos(k_0 r) + \frac{\kappa}{k_0}\sin(k_0 r)\right]\exp(-\kappa r). \tag{17}$$

This is the SPAC expression for 1-D attenuating media. Comparing with eq. (11), a sine function term appears in addition to an exponentially damping term. This means that not only amplitude but also phase of the normalized cross spectrum are modified by including attenuation. Taking a limit of $\kappa \to 0$, it is possible to reproduce eq. (11). This confirms that eq. (17) is an extension of the SPAC expression to attenuating media. I also point out that part of this expression was derived for a kicked damped oscillator (Snieder *et al.* 2010).

The derivation for 3-D cases is also rather straightforward. Green's function in space-frequency domain is

$$G(\mathbf{r},\omega) = -\frac{\exp(i(k_0+i\kappa)r)}{4\pi r} = -\frac{\exp(ik_0 r)}{4\pi r}\exp(-\kappa r), \tag{18}$$

and its imaginary part is

$$\mathrm{Im}[G(\mathbf{r},\omega)] = -\frac{\sin(k_0 r)}{4\pi r}\exp(-\kappa r). \tag{19}$$

Acknowledging the following relation

$$\lim_{\mathbf{r}\to 0}\mathrm{Im}[G(\mathbf{r},\omega)] = -\lim_{r\to 0}\frac{\sin(k_0 r)}{4\pi r}\exp(-\kappa r) = -\frac{k_0}{4\pi}, \tag{20}$$

and substituting this into eq. (4) leads to

$$C_{1,2}(\mathbf{r},\omega) = \frac{\sin k_0 r}{k_0 r}\exp(-\kappa r). \tag{21}$$

This is the SPAC expression for 3-D attenuating media. Comparing with eq. (13), only an exponentially damping term is added. This is different from 1-D cases. Taking a limit of $\kappa \to 0$, it is possible to reproduce eq. (13). This is also a confirmation that eq. (21) is an extension of the SPAC expression to attenuating media.

The derivation for 2-D cases is more complicated. Green's function in space-frequency domain is

$$G(\mathbf{r},\omega) = -\frac{iH_0^{(1)}((k_0+i\kappa)r)}{4}, \tag{22}$$

and its imaginary part is

$$\mathrm{Im}[G(\mathbf{r},\omega)] = -\frac{1}{4}\mathrm{Im}\left[iH_0^{(1)}((k_0+i\kappa)r)\right]. \tag{23}$$

Therefore, the following relation is obtained formally:

$$C_{1,2}(\mathbf{r},\omega) = \frac{\mathrm{Im}\left[iH_0^{(1)}((k_0+i\kappa)r)\right]}{\lim_{r\to 0}\mathrm{Im}\left[iH_0^{(1)}((k_0+i\kappa)r)\right]}. \tag{24}$$

However, some further manipulation is necessary to understand this relation better. Here, I analyse this relation by imposing a constraint of $k_0 \gg \kappa$ such that attenuation is small. For the case, multiplication formula of Hankel function (eq. 9.1.74 in Abramowitz & Stegun 1970) leads to

$$\begin{aligned} &H_0^{(1)}((k_0+i\kappa)r) \\ &\quad = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\left(1+i\frac{\kappa}{k_0}\right)^2 - 1\right]^n\left(\frac{k_0 r}{2}\right)^n H_n^{(1)}(k_0 r) \\ &\quad \approx \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\left(1+2i\frac{\kappa}{k_0}\right) - 1\right]^n\left(\frac{k_0 r}{2}\right)^n H_n^{(1)}(k_0 r) \\ &\quad = \sum_{n=0}^{\infty}\frac{(-i\kappa r)^n}{n!}H_n^{(1)}(k_0 r). \end{aligned} \tag{25}$$

Truncating the expansion at the first order of $(\kappa/k_0)$, imaginary part of the Green's function is

$$\mathrm{Im}G_0(\mathbf{r},\omega) \approx -\frac{1}{4}\left[J_0(k_0 r) + \left(\frac{\kappa}{k_0}\right)(k_0 r)Y_1(k_0 r)\right], \tag{26}$$

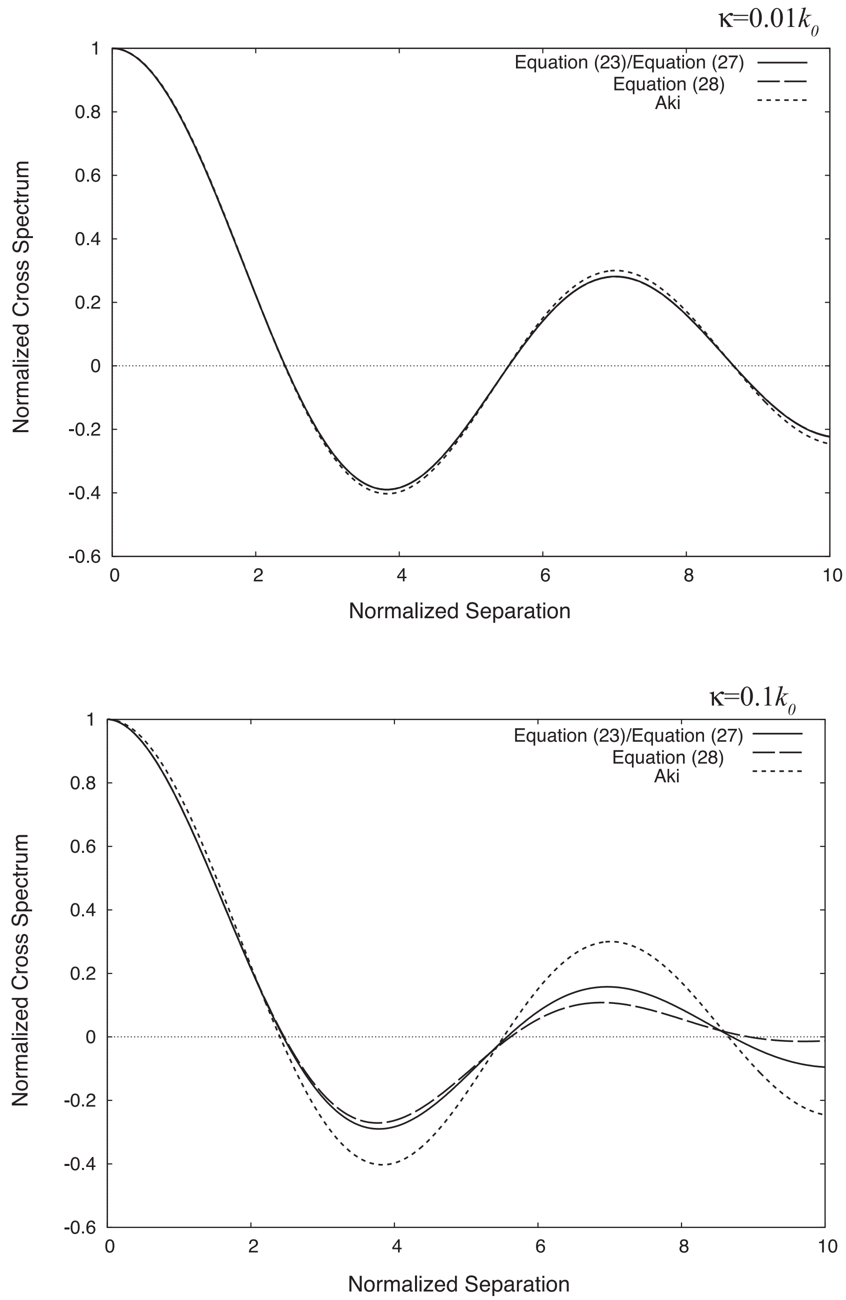



**Figure 2.** A strict SPAC expression in 2-D case (eq. (23) divided by eq. (27); solid curve), an approximation for small attenuation (eq. (28); broken curve) and Aki's (1957) original relation without dissipation (dotted curve). Top is for $(\kappa/k_0) = 0.01$ and bottom is for $(\kappa/k_0) = 0.1$.

where $Y_1$ means the first order Bessel function of the second kind or Neumann function. Taking the limit of $r \to 0$,

$$\lim_{\mathbf{r}\to 0} [\mathrm{Im} G_0(\mathbf{r}, \omega)] \approx -\frac{1}{4}\left[1 - \frac{2}{\pi}\left(\frac{\kappa}{k_0}\right)\right]. \quad (27)$$

Therefore, the following relation can be derived by substituting eqs (26) and (27) into eq. (4) and by leaving terms up to the first order of $(\kappa/k_0)$,

$$C_{1,2}(\mathbf{r}, \omega) \approx J_0(k_0 r)\left[1 + \frac{2}{\pi}\left(\frac{\kappa}{k_0}\right)\right] + \left(\frac{\kappa}{k_0}\right)(k_0 r) Y_1(k_0 r). \quad (28)$$

This is an approximated SPAC expression for 2-D attenuating media under a constraint of small attenuation. Comparing with eq. (12), an additional term of Neumann function appears.

In Fig. 2, eq. (28) is shown by broken curves for $(\kappa/k_0) = 0.01$ and $(\kappa/k_0) = 0.1$ to see its validity. For comparison, a strict expression which is eq. (23) divided by eq. (27) is shown by a solid curve, and eq. (12) by a dotted curve. For $(\kappa/k_0) = 0.01$, the strict expression and eq. (28) overlap with each other showing the validity of eq. (28). Reasonably, the two expressions decay faster with normalized separation $k_0 r$ than eq. (12). For $(\kappa/k_0) = 0.1$, eq. (28) decays more quickly with the normalized separation than the strict expression, which means that eq. (28) derived for small attenuation no longer holds. But interestingly, the phase differences between the three expressions seem not so large that troughs, peaks and zeros on the plot roughly match among the curves. Taking a limit of $\kappa \to 0$ in eq. (28), it is possible to reproduce eq. (12). This confirms that eq. (28) is an extension of the SPAC expression to media with small attenuation.

I can also analyse eq. (24) under another assumption of a large separation distance of $k_0 r \gg 1$. Using an asymptotic expansion of Hankel function for a large argument (eq. 9.2.3 in Abramowitz & Stegun 1970) and retaining up to the first-order terms of $(\kappa/k_0)$,

$$\mathrm{Im}G_0(\mathbf{r},\omega) \approx -\frac{1}{4}\exp(-\kappa r)\left[J_0(k_0 r) + \frac{1}{2}\left(\frac{\kappa}{k_0}\right)Y_0(k_0 r)\right]. \quad (29)$$

Putting eqs (29) and (27) into eq. (4) and leaving the terms up to the first order,

$$C_{1,2}(\mathbf{r},\omega) \approx \exp(-\kappa r)\left[J_0(k_0 r)\left(1 + \frac{2\kappa}{\pi k_0}\right) + \frac{\kappa}{2k_0}Y_0(k_0 r)\right]. \quad (30)$$

This is another form of the SPAC expression for 2-D attenuating media under the constraint of large separation distances and small attenuation. Taking a limit of $\kappa \to 0$ similarly, it is possible to reproduce eq. (12).

Here, the relation conjectured by Prieto *et al.* (2009) is reminded:

$$C_{1,2}(\mathbf{r},\omega) = J_0(k_0 r)\exp(-\kappa r). \quad (31)$$

They obtained this relation by multiplying eq. (12) of the SPAC expression for non-attenuating media with the exponentially damping term. Here, eq. (31) is shown to be obtained by taking a limit of $(\kappa/k_0) \to 0$ in [ ] of eq. (30). Therefore, eq. (31) turns out to be an approximated form of the SPAC expression for 2-D attenuating media under assumptions of large separation distances and very weak attenuation.

In Fig. 3, I consider the validity of eqs (30) and (31) for $(\kappa/k_0) = 0.01$and $(\kappa/k_0) = 0.1$. A strict expression, which is eq. (23) divided by eq. (27), is shown by a solid curve. Eqs (30) and (31) are shown by broken and dotted curves, respectively. Eq. (30) almost matches the strict expression for both values of $\kappa/k_0$ except for around zero in the normalized separation. This is because $Y_0$ term in eq. (30) tends to negative infinity around zero. But eq. (30) is thought to be better over wider range of the normalized separation, because this is originally derived for large separation. On the other hand, the conjecture by Prieto *et al.* (2009) of eq. (31) shows a good match with the strict expression for $(\kappa/k_0) = 0.01$ but a slight difference for $(\kappa/k_0) = 0.1$. But the difference is at most 5 per cent in amplitude. Therefore, I judge that Prieto *et al.* (2009) gives a simple but good approximation for weak attenuation.

## 3 DISCUSSION

The SPAC expressions are strictly derived in this study for infinite homogeneous media having spatially uniform velocity and attenuation. First, I discuss whether or not the SPAC expressions are applied to more realistic heterogeneous media. Seismic interferometry is shown to hold for infinite media having heterogeneous velocity but uniform attenuation as far as the generalized optical theorem holds (Margerin & Sato 2011). For the case, homogeneous noise distribution is necessary to compensate dissipation because of homogeneous attenuation. The similarity between seismic interferometry and the SPAC method suggests that the SPAC expressions are also applicable to the media. Even for heterogeneous distribution of attenuation, Snieder (2007) showed that seismic interferometry holds. But noise sources must be distributed so as to balance exactly with the heterogeneous attenuation. Cupillard & Capdeville (2010) showed by numerical simulations that noise sources distributed in a patch non-uniformly made it difficult to estimate attenuation by seismic interferometry. The SPAC method may partly reduce the effect of heterogeneous distribution of noise sources and attenuation by taking average over many station pairs having similar separations but different azimuth angles, but cannot remove the effect totally. Here it is reminded that an ensemble average in eq. (4) for dissipative media mathematically corresponds to a volume integral with respect to the noise sources, which contrasts with a surface integral for non-dissipative media. Therefore, it seems safe so far to apply the SPAC expressions in regions where heterogeneities in attenuation and noise source distribution are not so strong.

The second point of discussion is frequency dependence of attenuation. Attenuation can be included by using a complex wavenumber. In this study, the generalized wave eq. (14) is considered. The attenuation term is proportional to the time derivative of the wavefield $u$ in this equation, and the complex wavenumber is expressed as:

$$k = k_0 + i\kappa. \quad (32)$$

If the imaginary part $\kappa$ is sufficiently smaller than the real part $k_0$, quality factor $Q$ is defined as

$$Q = \frac{k_0}{2\kappa} = \frac{\omega}{2\kappa V_0}. \quad (33)$$

(e.g. Aki & Richards 2002). Because $k_0 = \omega/V_0$ and $\kappa$ is constant, $Q$ is found to be linearly proportional to frequency for this case. This type of attenuation is easiest to handle, because amplitude decays with waveform undistorted. This is the principal reason why I used eq. (14). On the other hand, Tsai (2011) dealt with another type of damped wave equation:

$$\Delta u(\mathbf{r},t) - \left(\frac{1}{V_0^2}\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2} + \frac{2\kappa}{V_0}\frac{\partial u(\mathbf{r},t)}{\partial t}\right) = 0. \quad (34)$$

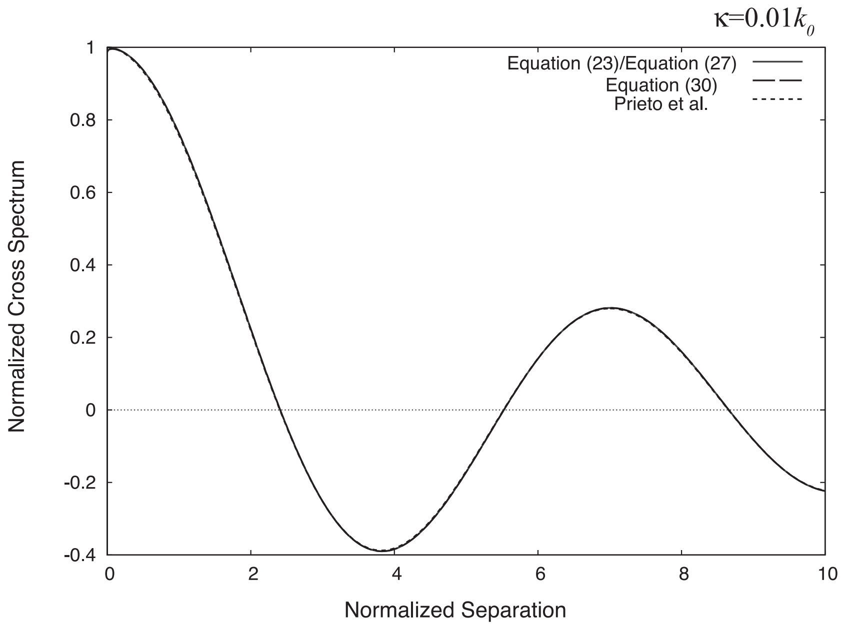
For this equation, the complex wavenumber is calculated to be:

$$k = k_0\sqrt{1 + \frac{2i\kappa}{k_0}}. \quad (35)$$

Expanding this up to the second order of $(\kappa/k_0)$ in Taylor series, $Q$ is estimated to be:

$$Q \approx \frac{\omega}{2\kappa V_0} + \frac{\kappa V_0}{4\omega}, \quad (36)$$

showing a more complicated dependence on frequency. Therefore, waveforms are distorted with propagation. But in any case, seismic interferometry holds true for this equation too, because this type of damped wave equation is also expressed by a linear partial differential equation (e.g. Snieder *et al.* 2007). And if the second term in the right hand side of eq. (36) is negligible to the first term, eq. (36) is similar to eq. (33). This suggests that solutions of eqs (14) are similar to solutions to eq. (34) for $(\kappa/k_0) \ll \sqrt{2}$. Roux *et al.* (2005) introduced a constant imaginary part of velocity with frequency, which is equivalent to constant $Q$ with frequency. The constant $Q$ explains why higher frequency components decay more quickly than lower frequency components in their result. The case of constant $Q$ is not easy to handle, probably because it is hard to express wave equation in a linear partial differential equation for the case.



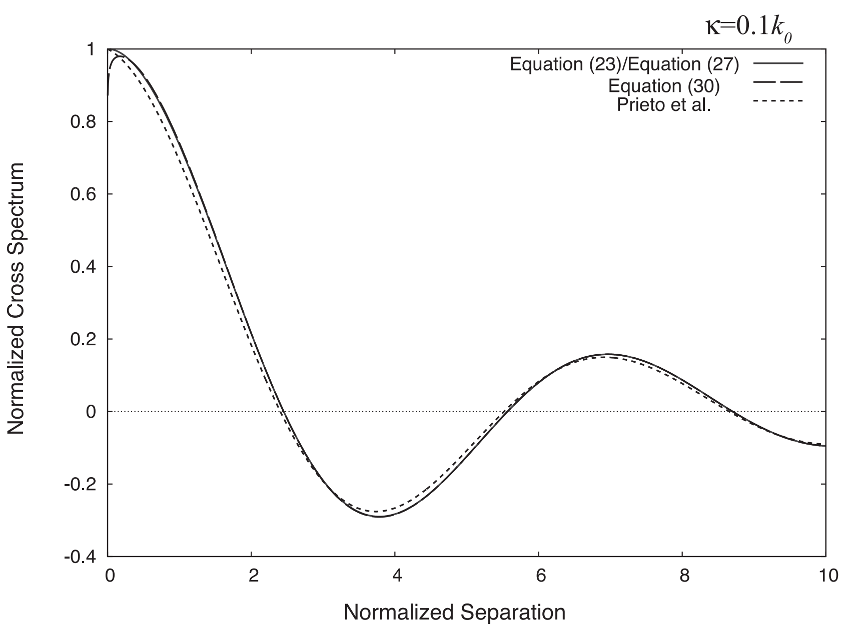




**Figure 3.** A strict SPAC expression in 2-D case (eq. (23) divided by eq. (27); solid curve), an approximation for small attenuation and larger station separation (eq. (30); broken curve) and the conjecture of Prieto *et al.* (2009) (eq. (31); dotted curve). Top is for $(\kappa/k_0) = 0.01$ and bottom is for $(\kappa/k_0) = 0.1$.

## 4 CONCLUSION

In this study, the SPAC method has been successfully formulated for 1-D, 2-D and 3-D scalar waves in homogeneous dissipative media having uniform velocity and attenuation based on the generalized wave equation or the generalized telegraph equation. The derivation is rather straightforward thanks to a close relation between the SPAC method and seismic interferometry. For 3-D case, the normalized cross spectrum for a dissipative medium is found to be a multiplication of the SPAC expression for a non-dissipative medium with an exponentially damping term. However, expressions for 1-D and 2-D cases are not as simple as 3-D cases, which means that not only amplitude but also phase in the normalized cross-spectrum are modified by the effect of dissipation. The SPAC expressions derived in this study seem to be applicable to media having rather homogeneous distribution of attenuation and noise sources. The conjecture of Prieto *et al.* (2009) has turned out to be not a strict solution but a good approximation for small dissipation. This study will form a theoretical foundation to estimate not only phase velocity but also attenuation from analysis of ambient noises.

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