ENERGY PROPAGATION INCLUDING
SCATTERING EFFECTS
SINGLE ISOTROPIC SCATTERING
APPROXIMATION

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The elastic energy propagation in a three dimensional infinite elastic medium, in which scatterers are distributed homogeneously and randomly, is investigated by a statistical method. A single isotropic scattering process is investigated. The elastic medium is characterized by the wave velocity \( V \) and the distribution of the scatterers is characterized by the mean free path \( l \). It is assumed that the elastic energy is radiated spherically from the source at a time \( t=0 \) in a short time duration. A space-time distribution of the mean energy density of the scattered waves is obtained as
\[
E_s(r, t) = \left( W_s/4\pi V t r \right) \ln \left( (Vt+r)/(Vt-r) \right) \text{ for } Vt \geq r,
\]
where \( r \) is the distance from the source and \( W_s \) is the total energy radiated. A uniform spatial distribution is constructed far behind the wave front and near the source. The mean energy density \( E_s \) is proportional to \( r^{-2} \) for \( r \equiv 2r/V \) and independent of \( r \) and \( W_s \). Several important properties of coda waves observed near the hypocenter are explained qualitatively by this solution when heterogeneities in the earth are interpreted as the scatterers and \( E_s \) corresponds to the power spectrum of coda waves.

1. Introduction

The object of this paper is to construct a simple model which explains properties of coda waves observed near the hypocenter. High frequency seismic waves are interesting because of their sensitivity to the earth's fine structures.

Aki and Chouet (1975) enumerated several important properties of coda waves as follows:

[A] "The study of coda by a small-aperture array of seismographs shows that they are not regular plane waves coming from the epicenter" (Aki et al., 1958; Aki and Tsujiura, 1959).

[B] "The power spectra of coda waves from different local earthquakes decay as a function of time (measured from the earthquake origin time) in the same manner independent of the distance and the nature of the path between the epicenter and station" (Aki, 1969; Aki and Chouet, 1975).
"The above time dependence [B] is also independent of the earthquake magnitude, at least for small earthquakes with $M < 6$" (Aki, 1969; Aki and Chouet, 1975).

For a given local earthquake at epicentral distance shorter than about 100 km the total duration of a seismogram $T_{c-p}$ (the length of time from the beginning of P waves to the end of coda) is nearly independent of the epicentral distance or azimuth and can be used effectively as a measure of the earthquake magnitude” (Bisztricsany, 1958; Soloviev, 1965; Tsumura, 1967).

Aki made a single back scattering model, in order to explain the above properties of coda waves. The heterogeneities in the earth medium were treated by a statistical method (Aki, 1969, 1973; Aki and Chouet, 1975). This model is based on the fact that the source time duration of earthquakes with $M < 6$ is less than a few seconds.

Then coda waves from small earthquakes were interpreted as the single back scattered waves from numerous heterogeneities distributed in the earth medium. The heterogeneities, origins of the scattering phenomenon, were considered as velocity anomalies which were characterized by their fluctuations and their correlation distance $a$. In the single back scattering model, the case for $\lambda \ll a$ (where $\lambda$ is the wavelength) which corresponds to the large forward scattering due to the shadow scattering was investigated. Only the time dependence of the mean energy density of the scattered waves at the source was derived.

Coda waves are considered not only as the scattered body waves but also as the scattered surface waves. Aki and Chouet investigated both cases.

Here, we notice that the effects of the multiple scattering were considered through a diffusion process (Aki and Chouet, 1975). The diffusion model had been applied previously to a problem of the seismic energy by Wesley (1965) and Nakamura et al. (1970, 1976).

In this paper, we investigate how the elastic energy propagates in the three-dimensional infinite elastic medium, in which numerous scatterers are distributed homogeneously and randomly, when the elastic energy is radiated spherically. That is, we restrict ourselves to the body wave scattering only. An isotropic scattering is assumed. Effects of the single scattering are considered in this approximation. Then we can derive a space-time distribution of the mean energy density of the single scattered waves. This model is one of generalizations of Aki’s single back scattering model. This simple model explains qualitatively all of the above-mentioned properties of coda waves from [A] to [D]. We restrict ourselves to construct a simple theoretical model and shall not analyze the observation data quantitatively. The space-time distribution is obtained in section 2, properties of the solution are discussed
in sections 3 and 4. Conclusions are summarized and discussed in section 5.

2. Single Isotropic Scattering Approximation

2.1 In elastic medium

We consider physical processes as follows. In a three dimensional infinite elastic medium, the elastic energy is radiated spherically from a point source during a short time interval \( t \). The point source assumption means that the size of the source is small compared with the wavelength \( \lambda \) of the emitted elastic wave. To simplify the problem, we assume that the medium is characterized by the mass density \( m \) and the wave propagation velocity \( V \). And we assume that the medium is non-dispersive and the elastic energy is not converted into heat in it. Then, the waves travel without deforming their shapes. Their amplitudes change only due to a geometrical spreading.

Let \( P_\nu(r, t|\omega) \) be a power spectrum of the \( \nu \)-component of the displacement vector of the elastic medium at a co-ordinate \( r \) determined from a short sample around a given time \( t \), where \( \omega \) is the angular frequency (Aki, 1969). Mean values, which appear in the following discussion, are determined from a short sample around a given time \( t \). The mean energy density of the elastic waves within a unit angular frequency band around \( \omega \) is denoted by \( E(r, t|\omega) \), which is defined as follows. For propagating plane waves the mean energy density is twice the mean kinetic energy density, so we have

\[
E(r, t|\omega) = \sum_{\nu=1}^{3} m \omega^2 P_\nu(r, t|\omega)
\]

(Aki and Chouet, 1975). Only the mean energy density is investigated in the following.

Let \( L_\omega(t|\omega) \) be the mean energy generation per unit time of the elastic wave within a unit angular frequency band around \( \omega \) at the source, where subscript \( u \) denotes the source time duration. The source is placed at the origin.

The mean energy flux density of the elastic wave within a unit angular frequency band around \( \omega \), \( J(r, t|\omega) \), is given at a time \( t \) with a co-ordinate \( r \) as

\[
J(r, t|\omega) = \frac{1}{4\pi r^3} L_\omega \left( t - \frac{r}{V} |\omega| \right) \frac{r}{r}
\]

where \( 1/4\pi r^3 \) is a geometrical spreading factor and \( r = |r| \).

In the wave zone \( (r \gg \lambda) \), a spherical wave can be considered as a plane wave in a small region of the space. Then the mean energy density \( E \) is related to \( J \) by

\[
J(r, t|\omega) = E(r, t|\omega) \cdot V \cdot \frac{r}{r}
\]
2.2 Distribution of the scatterers

Now, we suppose that scatterers are distributed randomly and homogeneously with a number density \( n \) in the elastic medium. Then, the scattered waves can be considered as incoherent waves.

Here, cracks, faults, density anomalies and velocity anomalies in the earth medium are considered as the scatterers.

The scattering is generally characterized by what is called the total effective cross section \( \sigma \), which is the ratio of the time average of the scattered wave energy per unit time to the mean energy flux density of the incident wave, and evidently has the dimension of area (Landau and Lifshitz, 1959). In general, the total effective cross section depends on the frequency of the incident wave and the size of the scatterer.

When scatterers are distributed homogeneously with the number density \( n \), the length \( l \equiv (n \sigma)^{-1} \) is usually called the mean free path and \( l/V \) the mean free time. The scatterers reduce the mean energy flux density of the incident plane wave by \( e^{-x/l} \), where \( x \) is the distance along the propagation direction.

Here, we notice that this exponential damping does not mean the dissipation of the energy but the energy transfer from the primary wave to the scattered waves. Thus, the mean free path is different from the \( Q \)-value, which reduce the whole energy of the elastic waves by \( e^{-\omega t/\alpha} \).

The turbidity may correspond to \( n \sigma \) (Aki and Chouet, 1975). The turbidity was given to be of the order of \( 10^{-7} \sim 10^{-8} \text{ cm}^{-1} \) at frequencies higher than 10 Hz (Aki and Chouet, 1975).

2.3 Isotropic scattering assumption

Here, we assume the isotropic scattering in order to obtain the analytic solution with simple calculations. It is well known that the forward scattering is large for \( \lambda \ll a \) due to the shadow scattering, where \( a \) is the size of the scatterer (Morse and Feshbach, 1953). In the opposite case, the back scattering is comparatively large for \( \lambda \gg a \) (Yamakawa, 1962; Morse and Feshbach, 1953). So, it seems that the rough isotropic scattering assumption corresponds to an intermediate condition. But, the angular dependence of the scattering process and the total effective cross section can not be derived without solving the boundary value problem in the elastic theory.

The conversion scatterings between the longitudinal waves and the transverse waves are not considered because of the simplified assumption; the medium is characterized by one wave velocity.

Here, we notice that \( \sigma \) depends on \( \omega \).

2.4 Single isotropic scattering approximation

Now, let us suppose that the mean free path \( l \) is much longer than the
distance $r$ under the consideration ($r \ll l$). Since the scattering is assumed to be a weak process, only the single scattering is considered ($t \ll l/V$). The single scattered waves are the lowest order of the scattered waves. In the single isotropic scattering approximation, scattering effects must be considered up to the first order of $\sigma$.

From the definition of the total effective cross section, the mean energy flux density $J_d$ is written as

$$J_d(r, t|\omega) = \frac{e^{-\sigma r}}{4\pi r^2} L_a \left( t - \frac{r}{V} \mid \omega \right) \frac{r}{r} \approx \frac{(1 - \nu \sigma r)}{4\pi r^2} L_a \left( t - \frac{r}{V} \mid \omega \right) \frac{r}{r},$$  \hspace{1cm} (4)

which includes the energy attenuation due to the scattering, and the subscript d denotes the direct wave. From Eq. (3) the mean energy density of the direct wave $E_d$ is written in the wave zone as

$$E_d(r, t|\omega) = \frac{(1 - \nu \sigma r)}{4\pi r^2} \frac{L_a \left( t - \frac{r}{V} \mid \omega \right)}{V}.$$  \hspace{1cm} (5)

The energy of the direct wave decreases monotonously and is transferred into the scattered waves.

We take the single scattered waves into consideration. The mean energy density $E$ can be expressed as the sum of $E_d$ and the mean energy density of the single scattered waves $E_s$ in the single isotropic scattering approximation.

We take the co-ordinate vector of the $i$-th scatterer $r_i$ and the observer $r$ as shown in Fig. 1. The mean energy flux density at $r_i$ is given by Eq. (4),

$$J_d(r_i, t|\omega) = \frac{(1 - \nu \sigma r_i)}{4\pi r_i^2} L_a \left( t - \frac{r_i}{V} \mid \omega \right) \frac{r_i}{r_i}. \hspace{1cm} (6)$$

The scattering is isotropic, so that it is equivalent to the spherical symmetric energy generation per unit time $\sigma |J_a|$ in the wave zone. The mean energy flux density, $J_s$, of the single isotropic scattered wave from the $i$-th scatterer at $r$ is given by

$$J_s(r, t|\omega) = \frac{(1 - \nu \sigma r_i)}{4\pi r^2} \sigma |J_a| \left( r_i - \frac{r_i}{V} \mid \omega \right) \frac{r_i}{r_i},$$ \hspace{1cm} (7)

![Fig. 1. Configuration of the source, the observer and the scatterers.](image)
where \( r' \equiv r - r_i \) and the subscript \( s \) denotes the single scattered waves. When
the observer is in the wave zone of the single scattered spherical wave, the
mean energy density of the single scattered wave from the \( i \)-th scatterer \( E_i \) is
given by
\[
E_i(r, t|\omega) = \frac{\sigma}{V(4\pi)^3 r_i^2 r_i^2} L_a\left( t - \frac{r_1 + r_3}{V} \right| \omega), \tag{8}
\]
where terms including the second order in \( \sigma \) are neglected. The summation
of all the \( E_i \)'s gives the mean energy density of the single scattered waves \( E_s \),
because \( E_i \) is the scalar density. The summation can be changed into an
integration because of the homogeneous distribution. The integrated value de-
deps on \( r = |r| \), so that we can rewrite \( E_s \) as
\[
E_s(r, t|\omega) = \frac{\nu \sigma}{(4\pi)^3 V} \int \int \int_{-\infty}^{\infty} \frac{1}{r_i^2 r_i^2} L_a\left( t - \frac{r_1 + r_3}{V} \right| \omega) d^3 x, \tag{9}
\]
where \( x \) is the co-ordinate vector of the scattering point, \( r_i = |x| \) and \( r_i =
|x - r| \). The integration is carried out all over the space, where \( r_1 \) corre-
sponds to \( r_i \) and \( r_3 \) to \( r_i \) in Eq. (8). Equation (9) is rewritten as
\[
E_s(r, t|\omega) = \nu \sigma \int_{-\infty}^{\infty} G_s(r, t - t') L_a(t'|\omega) dt'. \tag{10}
\]
Here, the Green function is defined by
\[
G_s(r, t) = \frac{1}{(4\pi)^3 V} \int \int \int_{-\infty}^{\infty} \frac{1}{r_i^2 r_i^2} \theta\left( t - \frac{r_1 + r_3}{V} \right) d^3 x. \tag{11}
\]
It is seen from the term \( \theta(t - (r_1 + r_3)/V) \) in Eq. (11) that the single scattered
waves at a distance \( r \) and at a time \( t \) consists of the scattered waves from the
scatterers on a spheroidal shell of which the foci are the source and the ob-
server.

The integration of Eq. (11) can be carried out easily, when the co-or-
dinates are transformed into the prolate spheroidal co-ordinates. This integra-
tion is given in Appendix-A.

The following equations are obtained;
\[
G_s(r, t) = \frac{1}{4\pi r^2} K\left( \frac{rt}{r} \right) \theta\left( \frac{rt}{r} - 1 \right), \tag{12}
\]
\[
K(x) = \frac{1}{x} \ln \frac{x + 1}{x - 1}, \tag{13}
\]
\[
\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}. \tag{14}
\]

The step function \( \theta \) in Eq. (12) corresponds to the causality condition which
means that the apparent propagation velocity is less than or equal to \( V \).
3. Properties of the Solution

In the preceding section, it is assumed that the elastic energy is generated at the source in the short time duration $u$. Now, we suppose that the energy generation is expressed by

$$L_\delta(t|\omega) = \frac{W_\delta(\omega)}{u} S_\delta(t) ,$$  \hfill (15)

$$S_\delta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq u \\ 0 & \text{for otherwise} \end{cases} ,$$  \hfill (16)

where $W_\delta(\omega)$ is the total radiated energy within a unit angular frequency band around $\omega$. Then $E_\delta$ is written as

$$E_\delta(r, t|\omega) = \frac{(1-nar)}{4\pi r^2} \frac{W_\delta}{V \cdot u} S_\delta \left( t - \frac{r}{V} \right) .$$  \hfill (17)

Similarly, $E_\sigma$ can be given by substituting Eq. (15) in Eq. (10).

In the preceding section we have treated each scatterer as a point. But the scattering does not occur in an instance because of the size of the scatterer. So, we shall restrict ourselves to the mean energy density of the scattered waves over the interval $\Delta T$, where $V \cdot \Delta T$ is larger than the size of the scatterer and also $\Delta T$ is larger than $u$.

We can treat $(1/u)S_\delta(t)$ as Dirac's delta function, then $E_\sigma$ can be given by

$$E_\sigma(r, t|\omega) = \frac{n\sigma W_\sigma}{4\pi r^2} K \left( \frac{V t}{r} \right) \theta \left( \frac{V t}{r} - 1 \right) .$$  \hfill (18)

The function $K$ diverges logarithmically as $V t / r \to 1$.

However, this is no serious trouble because $E_\sigma$ is not the energy itself but the energy density. As shown in Appendix-B, the integration of $E_\sigma$ all over the space is finite. This logarithmic divergence occurs at $r_1=0$ and $r_2=0$. These points are out of the region of our approximation, because $r_1$ and $r_2$ are assumed to be much longer than the wavelength $\lambda$.

Let us put

$$\rho \equiv nar , \quad \tau \equiv n\sigma V t , \quad \mu \equiv n\sigma V u .$$  \hfill (19)

Here, we define the following dimensionless quantities

$$\varepsilon_\delta \equiv \frac{E_\delta}{(n\sigma)^8 W_0} ,$$  \hfill (20.1)

$$\varepsilon_\sigma \equiv \frac{E_\sigma}{(n\sigma)^8 W_0} .$$  \hfill (20.2)

The dimensionless quantities $\varepsilon_\delta$ and $\varepsilon_\sigma$ are investigated instead of $E_\delta$ and $E_\sigma$. From Eqs. (17) and (18) the following equations are given;
\[\varepsilon_s(\rho, \tau) = \frac{(1-\rho)}{4\pi\rho^2} S_\rho(\tau-\rho), \quad (21.1)\]

\[\varepsilon_s(\rho, \tau) = \frac{1}{4\pi\rho^2} K\left(\frac{\tau}{\rho}\right) \theta\left(\frac{\tau}{\rho} - 1\right). \quad (21.2)\]

The argument \(\tau/\rho\) means the time normalized by the travel time \((t_0 = r/V)\) at \(r\) and it is equal to \(t/t_0\). We can rewrite Eq. (13) as

\[K\left(\frac{\tau}{\rho}\right) = \frac{\theta}{\tau} \ln \frac{\tau + \rho}{\tau - \rho} = \frac{t_0}{t} \ln \frac{t + t_0}{t - t_0}. \quad (22)\]

We show the function \(K\) in Figs. 2 and 3, where the argument \(\alpha\) is equal to \(\tau/\rho\). For \(\tau \gg \rho\) \((t \gg t_0)\) the function \(K\) behaves as

\[K\left(\frac{\tau}{\rho}\right) \approx 2\left(\frac{\rho}{\tau}\right)^2 \approx 2\left(\frac{t_0}{t}\right)^2, \quad (23)\]

which is shown by the dotted curves in Figs. 2 and 3. Equation (23) means that the mean energy density of the single scattered waves decreases as \(r^{-2}\). From Figs. 2 and 3 this property holds good for \(\tau \geq 2\rho\) \((t \geq 2t_0)\).

Especially, the mean energy density of the single scattered waves at the source is given as follows;

\[E_s(0, t|\omega) = \lim_{\tau \to 0} E_s(r, t|\omega) = \frac{n\sigma W_0}{2\pi V^2 t^2} = \frac{W_0}{2\pi V^2 l^2 t^2}, \quad (24)\]

\[\varepsilon_s(0, \tau) = \frac{1}{2\pi t^2}. \quad (25)\]

\[K(\alpha) = \frac{1}{\alpha} \ln \frac{\alpha + 1}{\alpha - 1}\]

Fig. 2. The solid curve represents the time dependent factor of the mean energy density \(K(\alpha)\) and the dotted curve \(2/\alpha^2\). \(\alpha = \tau/\rho = t/t_0\), where \(t_0 = r/V\) is the travel time of the direct wave.
Energy Propagation Including Scattering Effects

![Graph](image)

Fig. 3. Same as the caption of Fig. 2.

In the single back scattering model, the above-mentioned time dependence for the body waves was derived previously by AKI and CHOUET (1975).

Next, we investigate a space-time distribution of the mean energy density. There remains the source time duration $\mu$ ($=n_0Vt$) as a free parameter, so that $\varepsilon_d$ and $\varepsilon_a$ cannot be shown in the same figure. We show $\mu \cdot \varepsilon_d$ and $\varepsilon_a$ in Figs. 4 and 5. In Fig. 4, the time variations of the mean energy density at different distances from the source are shown. In Fig. 5, the space distributions of the mean energy density are shown for fixed times.

From these figures, the mean energy density has a wave front at $\tau = \rho$ ($t = r/V$) and it decreases as the time increases. The spatial distribution is

![Graph](image)

Fig. 4. The time variation of the mean energy density at $\rho$ ($=n_0r$). The open circles represent $\mu \cdot \varepsilon_d$ and the vertical lines are the wave fronts. The solid curves represent $\varepsilon_a$ and the dotted curves represent $\varepsilon_a(0, \tau)$, where $\tau = n_0Vt$. 

almost flat far behind the wave front as well as near the source. This is the energy pool, which AKI and CHOUET referred to in their paper (1975). The energy pool is constructed even in the single isotropic scattering approximation.

4. Uniform Spatial Distribution of the Mean Energy Density

In this section, the energy pool is investigated in detail. It is seen from Fig. 4 that the mean energy densities at different distances from the source have different values near the fronts, but they decrease in the same manner and become close to each other as \( t \) increases. It is also seen from the fact that the curves in Fig. 5 are nearly flat sufficiently far behind the fronts.

We want to know how much time it takes before the mean energy densities become sufficiently close to each other for different \( r \). We show this aspect by comparing the mean energy density at a distance \( r \) from the source with the one at the source. Let us define the energy density ratio in the dimensionless form as

\[
f(\rho, \tau) = \frac{\varepsilon_\rho(\rho, \tau)}{\varepsilon_\rho(0, \tau)}.
\]

From Eqs. (21.2) and (25), the ratio \( f \) becomes a function of the ratio \( \tau/\rho \).

\[
f\left(\frac{\tau}{\rho}\right) = \frac{\tau}{2\rho} \ln \frac{\tau + \rho}{\tau - \rho} = \frac{Vt}{2r} \ln \frac{Vt + r}{Vt - r}.
\]

For \( \rho \ll \tau \) (\( r \ll Vt \)) the ratio \( f \) behaves as

\[
f\left(\frac{\tau}{\rho}\right) \approx 1 + \frac{1}{3} \left(\frac{\rho}{\tau}\right)^2 = 1 + \frac{1}{3} \left(\frac{r}{Vt}\right)^2.
\]
From Eq. (28) the energy density ratio \( f \) tends to 1 as \( \tau/\rho \to \infty \). We show this function \( f \) in Fig. 6, where the argument \( \alpha \) is equal to \( \tau/\rho \). The energy density ratio \( f \) is about 1.1 for \( \tau/\rho = 2 \) (\( r = Vt/2 \)). It means that the time dependence of the mean energy density near the source is nearly independent of \( r \) for \( r \ll Vt/2 \) and represented by Eq. (24). This \( r \) independence supports the fact that Aki's single back scattering model is a good approximation even for \( r \neq 0 \) when the isotropic scattering is assumed. From Eq. (24) the mean energy density is only proportional to \( W_0 \). And the time dependence is independent of \( W_0 \).

In the single isotropic scattering approximation, the above properties suggest that the power spectra of coda waves observed at different stations near the hypocenter decay as a function of time (measured from the earthquake origin time) in the same manner independent of the hypocentral distance. And the time dependence is also independent of the earthquake magnitude.

Here, we define the length of time \( T_{t-o} \), from the origin time to the end of coda. The end of coda is determined by the noise level at the observation point. We assume that the noise level is constant everywhere. Let \( E_s(\omega) \) be the mean energy density of the noise within a unit angular frequency band around \( \omega \). The length of time \( T_{t-o} \) is determined by the total radiated energy \( W_0(\omega) \) independent of \( r \) when the hypocentral distance is short. From Eq. (24) we obtain

\[
E_s(\omega) = \frac{W_0(\omega)}{2\pi \rho V^2 l(\omega) T_{t-o}^2}.
\]

(29)

\[
T_{t-o} = T_{t-p} + t_0,
\]

(30)
where $t_0$ is the travel time. When $t_0 \ll T_{r-o}$, we can rewrite Eq. (29) as

$$E_n(\omega) = \frac{W_d(\omega)}{2\pi V^2 l(\omega) T_{r-p}^2}. \quad (31)$$

The relation between $T_{r-p}$ and the total radiated energy $W_d(\omega)$ is given by Eq. (31). It is well known that the earthquake magnitude and $W_d(\omega)$ are related through the source spectrum of the seismic waves. When the $Q$-value is considered, Eq. (31) can be rewritten as

$$E_n(\omega) = \frac{W_d(\omega)e^{-\omega T_{r-p}/Q(\omega)}}{2\pi V^2 l(\omega) T_{r-p}^2}. \quad (32)$$

Equations (31) and (32) show that $T_{r-p}$ determines the earthquake magnitude qualitatively in the single isotropic scattering approximation.

5 Conclusions and Discussions

The results obtained are summarized here. Near the hypocenter, coda waves have been studied from the point of view of the scattering theory. Heterogeneities in the earth medium have been considered as the scatterers and they are treated by the statistical method.

The coda waves are interpreted as the single isotropic scattered body waves. This interpretation is consistent with property [A]. Then, we have obtained the space-time distribution of the mean energy density of the seismic (elastic) waves. This solution shows property [C]. This solution has a uniform spatial distribution far behind the wave front and near the hypocenter. This uniform spatial distribution, the energy pool, explains property [B]. Aki’s single back scattering model is justified from the single isotropic scattering approximation. And also this solution gives a qualitative relation between $T_{r-p}$ and the earthquake magnitude, which is property [D].

Thus, the single isotropic scattering approximation explains the several important properties of coda waves qualitatively.

Next, we discuss the simplified assumptions which we have set. The medium has been characterized simply by one wave velocity. When the medium is characterized by two wave velocities as the real elastic material, we have to consider the conversion scatterings between the longitudinal waves and the transverse waves. These conversion scatterings shall be investigated in the next paper in preparation.

We have assumed the isotropic scattering. It is an oversimplified assumption. The angular dependence of the scattering process for various types of the scatterers should be studied in detail. And it remains to calculate the space-time distribution of the mean energy density of the single scattered waves for the non-isotropic scattering.
We have studied only the single scattering process. When $T_{r-o} \approx l/V$, we have to investigate the double scattering process, the triple scattering process, ... and so on. It remains to study the multiple scattering process.

When we analyze the observation data quantitatively, we must consider the $Q$-value, the radiation pattern, effects of the earth's surface and the layered structure of the earth's crust, the surface waves and so on.

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Appendix-A

Green's function of the single isotropic scattering approximation is given by

$$G_s(r, t) = \frac{1}{(4\pi)^3 V} \int \int \int_{-\infty}^{\infty} \frac{1}{r_1 \cdot r_2} \delta \left( t - \frac{r_1 + r_2}{V} \right) d^3x.$$  \hspace{1cm} (A.1)

This integration is easily carried out, when the co-ordinates are transformed into the prolate-spheroidal co-ordinates. Let us place the source and the observer at the foci of the prolate-spheroid. The foci are on the $z$-axis and their co-ordinates are $(0, 0, C)$ and $(0, 0, -C)$ as shown in Fig. 7 (Morse and Feshbach, 1953).

The co-ordinate transformation is given by

$$\begin{align*}
x &= C \sqrt{\alpha^2 - 1} \sin \eta \cos \zeta, \\
y &= C \sqrt{\alpha^2 - 1} \sin \eta \sin \zeta, \\
z &= C \alpha \cos \eta,
\end{align*}$$  \hspace{1cm} (A.2)

where the domains of the variables are $1 \leq \alpha < \infty$, $0 \leq \eta \leq \pi$ and $0 \leq \zeta < 2\pi$. Then, the following relations are obtained,

Fig. 7. The source and the observer are placed at the foci of the prolate-spheroid.
\[ r_1 = \sqrt{x^2+y^2+(z+C)^2} = C(\alpha + \cos \gamma), \]
\[ r_2 = \sqrt{x^2+y^2+(z-C)^2} = C(\alpha - \cos \gamma), \]
\[ r_1 + r_2 = 2C \alpha. \]

(A.3)

The Jacobian is easily calculated as
\[ \text{det} = C'(\alpha^2 - \cos^2 \gamma) \sin \gamma \text{d}a \text{d}r \text{d}\zeta. \]

(A.4)

Equation (A.1) can be expressed as follows using Eqs. (A.2), (A.3) and (A.4);

\[
G_s(2C, t) = \frac{1}{(4\pi)^3 V} \int \int \int \frac{1}{C(\alpha^2 - \cos^2 \gamma)} \delta \left( t - \frac{2C \alpha}{V} \right) \sin \gamma \text{d}a \text{d}r \text{d}\zeta
\]

\[
= \frac{1}{16\pi C^2} \int_1^\infty \text{d}a \delta \left( \alpha - \frac{Vt}{2C} \right) \frac{1}{2\pi} \int_0^{2\pi} \text{d}\zeta \int_0^\pi \frac{\sin \gamma}{\alpha^2 - \cos^2 \gamma} \text{d}\gamma
\]

\[
= \frac{1}{4\pi(2C)^2} \int_0^\infty \text{d}a \delta \left( \alpha - \frac{Vt}{2C} \right) K(\alpha), \quad \text{(A.5)}
\]

where
\[
K(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \text{d}\zeta \int_0^\pi \frac{\sin \gamma}{\alpha^2 - \cos^2 \gamma} \text{d}\gamma = \frac{1}{\alpha} \ln \frac{\alpha + 1}{\alpha - 1}. \quad \text{(A.6)}
\]

Here, we define the step function as
\[
\theta(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{for } x < 0.
\end{cases} \quad \text{(A.7)}
\]

Then, we have the following equation, substituting Eqs. (A.6) and (A.7) into (A.5);

\[
G_s(r, t) = \frac{1}{4\pi r^2} K \left( \frac{Vt}{r} \right) \theta \left( \frac{Vt}{r} - 1 \right). \quad \text{(A.8)}
\]

Appendix-B. Energy Conservation Law

The energy conservation law is examined for theoretical consistency.

Let \( W_d(t|\omega) \) be the energy of the direct waves within a unit angular frequency band around \( \omega \) and \( W_s(t|\omega) \) the energy of the scattered waves. They are given as

\[
W_d(t|\omega) = \int_0^\infty E_d(r, t|\omega) 4\pi r^2 \text{d}r, \quad \text{(B.1)}
\]

\[
W_s(t|\omega) = \int_0^\infty E_s(r, t|\omega) 4\pi r^2 \text{d}r. \quad \text{(B.2)}
\]

These integrations can be carried out easily. We obtain the following equations.

\[
W_d(t|\omega) = W_d(\omega)(1 - n\sigma Vt), \quad \text{(B.3)}
\]

\[
W_s(t|\omega) = W_s(\omega)n\sigma Vt, \quad \text{(B.4)}
\]
where \((1/\mu)S_\delta(t)\) is considered to be Dirac's delta function. The sum of \(W_\delta\) and \(W_s\) gives the total energy \(W(t|\omega)\) in this approximation.

\[
W(t|\omega) = W_\delta(\omega) .
\]  

From Eq. (B.5), the energy conservation law is satisfied in the order of the approximation. The energy is conserved through the single scattering process.

REFERENCES


