THREE-DIMENSIONAL ANALYSIS OF SCATTERED P WAVES ON THE BASIS OF THE PP SINGLE ISOTROPIC SCATTERING MODEL

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Wave trains immediately following the initial motion of P waves are analyzed assuming that they are scattered P waves by heterogeneities in the earth medium. Three-dimensional partition of the energy density of scattered P waves is theoretically studied, where PP single isotropic scattering and homogeneous and random distribution of scatterers are assumed. The theoretical analysis shows that the three-dimensional mean trajectory for scattered P waves in velocity space is represented by a spheroid, which is prolate to the propagation direction of the direct P wave, and the spheroid tends to become spherical as time goes on. This model is applied to an analysis of short-period (1–30 Hz) records of small local earthquakes observed by a three-component velocity seismometer installed at the bottom of a deep borehole. The three-dimensional partition of energy density is calculated by using the covariance tensor for particle velocity observed. The observed horizontal component of energy density of scattered P waves agrees well with the theoretical one, although the observed vertical component is larger than and the observed radial component is smaller than the theoretical one. Our theoretical model is proved to be good in accounting for horizontal component. In order to explain changes in vertical plane, it is suggested that refraction and reflection at horizontal lateral boundaries and the earth's surface, and/or PS conversion scattering should be considered.

1. Introduction

Wave trains just following the initial motion of P waves can be interpreted as scattered P waves which are related to heterogeneities in the earth medium (cracks, faults, elastic moduli and/or density anomalies, etc.). The temporal change in the energy density of the scattered seismic body waves was theoretically studied on the basis of the single isotropic scattering model including PS and SP wave conversions by Sato (1977b). Three-dimensional partition of the energy density of seismic body waves was phenomenologically studied by Matsumura (1980) for the Kanto plain, Japan. He found the regional difference in the three-dimensional partition of the energy density.

In this paper, we first formulate the three-dimensional partition of the energy density of the scattered P waves on the basis of the single isotropic scattering model,
where we assume only PP scattering. Next, the theory is applied to the records of small local earthquakes observed by the three-component seismometer which is installed at the bottom of a borehole with a depth of 3,510 m.

2. Theory

We assume the spherically symmetric radiation for P waves, and let $L^p(t|\omega)$ be the energy radiation at the source per unit time within a unit angular frequency band around $\omega$ at time $t$. We also assume that the scattering is isotropic and is characterized by the total cross section $\sigma^{pp}(\omega)$ taking only PP scattering into consideration. And also, we suppose that the point-like scatterers are distributed homogeneously and randomly with a number density of $n^{pp}$, so that the scattered waves are mutually incoherent. In the following, we refer to this approximation as the PP single isotropic scattering model.

The energy density of the scattered P waves within a unit angular frequency band around $\omega$ at time $t$ is written as

$$E^{pp}_s(r,t|\omega) = \sum_{\nu=1}^3 E^{pp}_s(r,t|\omega),$$

(1)

where $E^{pp}_s$ is the $\nu$-component energy density of the scattered P waves, $r$ is the distance between the source and the observer, and the subscript $s$ means the scattered waves. Rigorously speaking, we cannot define the energy density at any angular frequency at any given time, but we can only define the averaged energy density at an angular frequency over several periods around a given time. The summation of the scattered waves from the homogeneously distributing scatterers is changed to the integral all over the space. The $\nu$-component energy density $E^{pp}_s$ is written as an integral;

$$E^{pp}_s(r,t|\omega) = n^{pp} \sigma^{pp}(\omega) \int_{-\infty}^{\infty} G^{pp}_{s,\nu}(r,t-t') L^p(t'|\omega) dt'.$$

(2)

The Green's function $G^{pp}_{s,\nu}$ is defined by

$$G^{pp}_{s,\nu}(r,t) = \frac{1}{(4\pi)^2 V^p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{k^2 \delta}{r_1 r_2} \left( t - \frac{r_1}{V^p} - \frac{r_2}{V^p} \right) d^2 y,$$

(3)

where $\delta$ is Dirac's delta function, $V^p$ the P-wave velocity, $\vec{y}$ the radius vector from the source to the scattering point, $r_1$ the distance from the source to the scattering point, $r_2$ the distance from the the scattering point to the observer, and $k_\nu$ the $\nu$-component of the unit vector $\vec{k} = \vec{r}_1 / r_1$ (see Fig. 1). Most of the notations follow SATO (1977b). We can easily evaluate the integral (3) by transforming the Cartesian coordinates to the prolate spheroidal coordinates (SATO, 1977a). The Green's functions defined by Eq. (3) are written as
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\[ \mathbf{r} = \hat{r} \mathbf{r}, \quad \mathbf{r}_s = \hat{y} \mathbf{r}, \quad \mathbf{r}_t = \hat{\mathbf{r}} - \hat{y} \mathbf{r} \]

Fig. 1. The vectors which define spatial relation among the source, the observer and the scatterer.

\[ G_{s,t}^{PP}(r,t) = G_{s,t}^{PP}(r,t) = \frac{1}{4\pi r^2} K_t \left( \frac{V_{pt}}{r} \right) \theta \left( \frac{V_{pt}}{r} - 1 \right), \]  \hspace{1cm} (4)

and

\[ G_{s,s}^{PP}(r,t) = \frac{1}{4\pi r^2} K_t \left( \frac{V_{pt}}{r} \right) \theta \left( \frac{V_{pt}}{r} - 1 \right), \] \hspace{1cm} (5)

where

\[ K_t(x) = \frac{x^2 + 1}{4x^2} - \frac{(x^2 - 1)^3}{8x^3} \ln \frac{x}{x - 1}, \] \hspace{1cm} (6)

and

\[ K_r(x) = \frac{x^2 + 1}{2x^2} + \frac{(x^2 + 1)^3}{4x^3} \ln \frac{x}{x - 1}, \] \hspace{1cm} (7)

and \( \theta \) is a step function. Subscripts \( t \) and \( r \) mean the transverse and radial components, respectively. The radial direction is defined by that connecting the source to the observer and the transverse direction is perpendicular to the radial direction. We take the first and the second axes as the transverse ones and the third axis as the radial one. Here, we note that

\[ 2K_t(x) + K_r(x) = \frac{1}{x} \ln \frac{x + 1}{x - 1} \equiv K(x) \] \hspace{1cm} (8)

and also

\[ \sum_{s=1}^{3} G_{s,s}^{PP}(r,t) = G_{s,s}^{PP}(r,t) \] \hspace{1cm} (9)

where \( K \) and \( G_{s,s}^{PP} \) are given in SATO (1977a).

The geometrical factor \( 4\pi r^2 \) is common to all of these Green’s functions. Therefore, the characteristics of the Green’s functions are easily seen by illustrating \( K_t(x) \) and \( K_r(x) \) as functions of \( x \), the time being normalized by the travel time of the direct P wave. Functions \( K_t(x) \), \( K_r(x) \) and \( K(x) \) are illustrated in Fig. 2, where the Poisson’s ratio is assumed to be 0.25. The arrival times of the initial P-phase and S-phase correspond to \( x = 1 \) and \( x = \sqrt{3} \), respectively. The function \( K_t(x) \) logarithmically diverging as \( x \to 1 \), however, the function \( K_r(x) \) tends to 0.5 as \( x \to 1 \). \( K_r(x) \) is always larger than \( K_t(x) \), while the difference between
them becomes smaller as \( x \) increases.

This model for scattered P waves indicates that the three-dimensional mean trajectory in the velocity space can be represented by a prolate spheroid, of which the aspect ratio is given by \( \sqrt{K_1(x)/K_2(x)} \). The prolate spheroid looks like a cigar for the initial stage of P-phase, however, it becomes spherical as time goes on.

3. Analysis

The \( \nu \)-component of the observed energy density within a unit angular frequency band around \( \omega \) at time \( t \) is given by

\[
E^{\text{obs}}_{\nu}(t|\omega) = [(1/2)\, m_{\nu}(t|\omega)^2 + \text{potential energy}]_{\text{av.}} = m[A_{\nu}(t|\omega)^2]_{\text{av.}}. \tag{10}
\]

where \( A_{\nu} \) is the \( \nu \)-component particle velocity and \( m \) the mass density. Suffix \( \text{av.} \) indicates an average value over several periods of particle motion. The total energy density is given by

\[
E^{\text{obs}}(t|\omega) = \sum_{\nu=1}^{8} E^{\text{obs}}_{\nu}(t|\omega). \tag{11}
\]

Direct comparison between the observed quantity \( E^{\text{obs}} \) and the theoretical one \( E^{\text{th}} \) is not practical, because the total P-waves' energy and the attenuation factor are not exactly known. As ratio \( E^{\text{obs}}_{\nu}/E^{\text{obs}} \) is independent of the total P-waves' energy and the attenuation factor, we compare observed ratio \( E^{\text{obs}}_{\nu}/E^{\text{obs}} \) with theoretical one \( E^{\text{th}}_{\nu}/E^{\text{th}} \). Let us compare \( E^{\text{obs}}_{1}/E^{\text{obs}} \) and \( E^{\text{obs}}_{2}/E^{\text{obs}} \) with \( K_1/K_2 \) and \( K_1/K \).

Seismic data for the analysis are obtained at the Iwatsuki observatory (35°
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Fig. 3. Over-all characteristic curves of the velocity seismometers (Iwatsuki observatory).

55°33' N, 139°44'17" E) which is located nearly in the center of the Kanto plain, Japan. A three-component velocity seismometer, of which natural frequency is 1 Hz, is installed at the bottom of a borehole with a depth of 3,510 m in the pretertiary formation (Takahashi and Hamada, 1975). Over-all characteristics of the instrument are illustrated in Fig. 3. Digitizing the analog data recorded on magnetic tapes at a sampling frequency of 62.5 Hz, we compute $E_{obs}(t|\omega)$ for a frequency band approximately ranging 1 to 30 Hz.

The original coordinate system indicating seismometric records is shown in Fig. 4, where Z-axis is vertical and X- and Y-axis are in the horizontal plane. A new coordinate system is chosen as follows; the radial axis (the third axis) is chosen as the maximum principal axis of the three-dimensional covariance

Fig. 4. X-, Y- and Z-axes: original coordinate system corresponding to the axes of the seismometers. Vertical [1], horizontal [2] and radial [3]-axes: new transformed coordinate system. The radial axis is chosen as the maximum principal axis of the covariance tensor calculated from the velocity amplitudes for a time-interval of 0.8 sec from the initial P-phase.
tensor calculated from the velocity amplitudes in a short time interval of 0.8 sec starting from the initial P-phase. The three-dimensional covariance tensor for the time interval of 0.8 sec is given by

\[
\sum_{i=1}^{\infty} \begin{bmatrix}
A_x(i \Delta t)A_x(i \Delta t) & A_x(i \Delta t)A_y(i \Delta t) & A_x(i \Delta t)A_z(i \Delta t) \\
A_y(i \Delta t)A_x(i \Delta t) & A_y(i \Delta t)A_y(i \Delta t) & A_y(i \Delta t)A_z(i \Delta t) \\
A_z(i \Delta t)A_x(i \Delta t) & A_z(i \Delta t)A_y(i \Delta t) & A_z(i \Delta t)A_z(i \Delta t)
\end{bmatrix}
\]

(12)

where the sampling interval \(\Delta t = 1/62.5\) sec.

Calculating experimentally the orientation of the maximum principal axis for

**Component**  **Short period seismograms**

- **Vertical**  
  ![Vertical Seismogram](image)

- **Horizontal**  
  ![Horizontal Seismogram](image)

- **Radial**  
  ![Radial Seismogram](image)

**Transformed**

- **Maximum principal axis**  
  ![Maximum principal axis](image)

**Original**

- **X-axis**  
  ![X-axis Seismogram](image)

- **Y-axis**  
  ![Y-axis Seismogram](image)

- **Z-axis**  
  ![Z-axis Seismogram](image)

**Covariance tensor**

Fig. 5. Seismograms in the original and the transformed coordinate systems. The covariance tensor is calculated for a time-interval of 0.8 sec from the initial P-phase.
time intervals of 0.4 and 1.6 sec, we find that the selection of time interval scarcely affects the orientation of the maximum principal axis. In the next place, we take the vertical axis (the first axis) in such a way that it is perpendicular to the radial one in the vertical plane. Meanwhile, the horizontal axis (the second axis) is also taken perpendicularly to the radial axis. The new coordinate system is also shown in Fig. 4. Examples of three-component seismograms in the original and the new coordinate systems are illustrated in Fig. 5. Reflected P waves from the surface near the borehole do not affect the determination of the maximum principal axis (the radial axis), because the reflected phase arrives 2.5 sec later than the initial P-phase at the bottom of the borehole. But, seismic waves of shallow earthquakes reflected at the surface near the epicenter are unavoidably included in the determination of the maximum principal axis.

Among 203 earthquakes observed during May, 1976, we choose 101 earthquakes, of which the deflection angle of the maximum principal axis from the azimuth of the epicenter is less than 30° in the horizontal plane. The hypocenters determined by the network of the Earthquake Research Institute, University of Tokyo (Tsumura, private communication, 1976) are used in this analysis. The epicenters, frequency distribution of S-P time and frequency distribution of earthquake magnitude are illustrated in Figs. 6 (a), (b), and (c), respectively.

After transforming the original coordinate system to the new one, we calculate each component of energy density $E_\phi$ for twenty time-intervals, of which the

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Fig. 6. (a) Epicenters of earthquakes used for the analysis (after Tsumura, private communication, 1976). (b) Frequency distribution of S-P time. (c) Frequency distribution of earthquake magnitude.
length is one-twentieth of S-P time. We illustrate the mean value of observed ratio $E_{\text{obs}}^{\text{obs}}/E_{\text{obs}}^{\text{obs}}$ for 101 earthquakes by solid circles and the standard deviation by length of vertical bar for each time-interval in Fig. 7. Theoretical curves $K_i/K$ and $K_r/K$ are also illustrated in Fig. 7, where $x$ is the time normalized by the travel time of direct P waves.

It appears that the horizontal component of observed ratio $E_{\text{obs}}^{\text{obs}}/E_{\text{obs}}^{\text{obs}}$ agrees well with theoretical curve $K_i/K$ for any $x$ from 1 to $\sqrt{3}$. However, the radial component of observed ratio $E_{\text{obs}}^{\text{obs}}/E_{\text{obs}}^{\text{obs}}$ is smaller than that given by theoretical curve $K_i/K$ and the vertical component of observed ratio $E_{\text{obs}}^{\text{obs}}/E_{\text{obs}}^{\text{obs}}$ is larger than theoretical $K_i/K$ for all $x$ from 1 to $\sqrt{3}$.

4. Discussion

Three-dimensional partition of energy density of scattered P waves is theoretically studied on the basis of PP single isotropic scattering model. Theoretical analysis predicts that the radial component of energy density is always larger than the transverse one and that the difference between them becomes smaller as time goes on. The theory is applied to an analysis of the records of seismic waves (1–30 Hz) observed by a three-component velocity seismometer installed
at the bottom of a deep borehole. We find a good agreement between the theory and the observation in the horizontal component. However, we find that the observed radial component of energy density is smaller than the theoretical one, and that the observed vertical component of energy density is larger than the theoretical one. We schematically illustrate the transverse and vertical mean trajectories for scattered P waves in velocity space in Figs. 8(a) and (b), respectively.

The above results demonstrate that the PP single isotropic scattering model provides a good approximation in the horizontal component, although the observational fact does not completely match our model. There are possibilities which qualitatively explain the discrepancy between the observed and theoretical results. One is PS conversion scattering we neglected, which possibly decreases the radial component of energy density. The others are refraction and reflection at horizontal lateral boundaries in velocity structure as well as the earth's surface. These effects may increase the vertical component of energy density rather than the horizontal component.

Our future study will include PS conversion scattering phenomena, in which polarization of scattered S waves will be important.

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