Amplitude attenuation of impulsive waves in random media based on travel time corrected mean wave formalism

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Waves gradually collapse with propagation through media with random velocity fluctuation; however, impulsive waves propagate without large attenuation when the wavelength is shorter than the correlation distance. The $Q^{-1}$ value predicted from the usual mean wave formalism monotonously increases with frequency even in the high-frequency limit, due to taking a mean over waves with large travel time fluctuations caused by the long scale velocity fluctuation compared with the wavelength studied. We propose a new statistical averaging method appropriate for the amplitude attenuation measurement of impulsive waves, in which the mean wave is defined after the correction of travel time fluctuations. We investigate impulsive scalar waves propagation in three-dimensional media with homogeneous and isotropic random fractional velocity fluctuation, based on the binary interaction approximation in this improved mean wave formalism. We successfully derive the $Q^{-1}$ value that has a peak of the order of the mean square fractional velocity fluctuation around the frequency corresponding to the correlation distance and decreases with frequency in the high-frequency limit, when the random media is characterized by the von Karman autocorrelation function. The correction of travel time fluctuation is shown to be equivalent to the neglect of energy scattering around the forward direction.

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INTRODUCTION

Waves generally collapse gradually with propagation through random velocity fluctuating media; however, impulsive waves propagate without large attenuation when the wavelength is shorter than the correlation distance. The problem of wave propagation in random media has been investigated extensively for decades: acoustic waves in turbulent gaseous15 and sea water with fluctuation caused by internal waves,6 electromagnetic waves,4,6 and elastic waves.5,7 In seismology, energy loss by scattering due to the random structure of the earth's lithosphere has been recently studied as a dominant mechanism for the amplitude attenuation of short period shear waves.6,8-12 which are considered to be impulsive waves. There are two fundamental statistical theories for the calculation of amplitude attenuation: the Born approximation1 and the binary interaction approximation in the mean wave formalism.5,6,8,9,10,12,13 Both theories predict that the $Q^{-1}$ value, which gives the damping of the wave amplitude per unit cycle as $e^{-\xi t}$, monotonously increases with frequency even in the high-frequency limit, in wavelengths shorter than the correlation distance of random media; however, such a frequency dependence of $Q^{-1}$ contradicts the fact that impulsive waves propagate without large attenuation in the high-frequency limit. Therefore we find that these statistical theories are not applicable to the amplitude attenuation of impulsive waves in random media, especially in the high-frequency limit.

In this paper, we will clarify the difference between the amplitude attenuation measurement of impulsive waves and the statistical averaging method of the usual mean wave formalism. Introducing a new statistical averaging method adequate for the amplitude attenuation measurement of impulsive waves, we will derive the frequency dependence of $Q^{-1}$ applicable for a wide range of frequencies based on the binary interaction approximation in the mean wave formalism. We will investigate scalar wave propagation through three-dimensional media with random fractional velocity fluctuation.

I. STATISTICAL AVERAGING METHOD APPROPRIATE FOR THE AMPLITUDE ATTENUATION MEASUREMENT OF IMPULSIVE WAVES

Scalar waves $\phi(x,t)$ at space coordinate $x$ and time $t$ in three-dimensional inhomogeneous media is governed by

$$\nabla^2 - 1/C^2 [1 - 2\xi(x)] |s|^2 \phi(x,t) = 0,$$  \hfill (1)

where $C$ is the mean velocity and $\xi(x)$ is the fractional velocity fluctuation. We assume that $\xi(x)$ is a homogeneous (stationary) and isotropic random function of space coordinate $x$.13 Let $\bar{\xi}$ be of the first-order small quantity ($\bar{\xi} \ll 1$). Now we imagine a family of random media, that is, an ensemble of $\xi(x)$. An ensemble average of $\xi$ must be zero:

$$\bar{\xi}(x) = 0,$$  \hfill (2)

where the overbar represents an ensemble average.

The autocorrelation function of the fractional velocity fluctuation $\xi$ is defined by

$$R(u) = \bar{\xi}(x+u)\bar{\xi}(x),$$  \hfill (3)

where $R$ is a function of $u = |u|$ because of the homogeneity and the isotropy property of $\xi(x)$. The power spectral density function of the fractional velocity fluctuation $\xi$ is defined by

$$P(m) = \int \int R(u)e^{-imu}du,$$  \hfill (4)

where $m$ is a wavenumber vector and $P$ is a function of $m = |m|$. Random media will be characterized by auto-
correlation function $R$ or power spectral density function $P$ rather than fluctuation $\xi$ itself.

The binary interaction approximation in the mean wave formalism is called the first-order smoothing method, which is equivalent to the simplest approximation based on the Dyson equation in the formal perturbation method.\(^7\) The validity of the mean wave formalism does not require any restriction on dimensions of the region occupied by the random inhomogeneities.\(^8\) Mean wave $\bar{\phi}$ is defined as an ensemble average of waves propagating through each medium after the generation under an identical initial condition. Taking an ensemble average of the wave equation directly, we obtain a stochastic equation governing the mean wave $\bar{\phi}$, where the effect of fluctuating wave $\phi'$ generated by the random fractional velocity fluctuation is included up to the second-order small quantity. We can easily obtain the $Q^{-1}$ value for the mean wave $\bar{\phi}$ as the integral over scattering angle $\vartheta$, from the imaginary part of the first-order perturbation solution of the dispersion relation for $\bar{\phi}$:

$$Q^{-1}(k) = \frac{k^2}{2\pi} \int_0^\pi P(2k \sin \vartheta/2) \sin \vartheta d\vartheta,$$

(5)

where $k$ is the wavenumber. This representation is equivalent to Eq. (27) of Karal and Keller.\(^9\) We can get the same representation for $Q^{-1}$ as (5) based on the Born approximation.\(^10\)

As an example, let us characterize random media by the exponential autocorrelation function with correlation distance $\sigma$ and the mean square fractional velocity fluctuation $\epsilon^2$. Then, $Q^{-1}$ increases with frequency of power $3$ for $ak < 1$, and it increases with frequency of positive power even for $ak > 1$. We imagine propagation of impulsive waves generated at the origin under an identical initial condition, where the duration time is supposed to be one or two times the characteristic period. We schematically illustrate the temporal changes in impulsive waves propagating through the random media for the low-frequency limit ($ak \ll 1$) and the high-frequency limit ($ak > 1$) in Figs. 1(a) and (b), respectively. If the fractional velocity fluctuation is small, the attenuation of the maximum amplitude in each experiment is also small as expected. For $ak \ll 1$, the travel time fluctuations are small compared to the wavelength studied, therefore the maximum amplitude of the mean wave $\bar{\phi}$ shows a small attenuation as illustrated at the bottom of Fig. 1(a). Travel time fluctuations become comparable to, or larger than, the wavelength $\lambda$ studied as $ak$ increases. In spite of the small amplitude attenuation in each experiment, the maximum amplitude of the mean wave $\bar{\phi}$ strongly decreases for $ak \gg 1$ as illustrated at the bottom of Fig. 1(b), because of the large travel time fluctuations.

In the amplitude attenuation measurement for impulsive waves, we are not taking an ensemble average of waves as schematically illustrated in Figs. 1(a) and (b). Our measurement may be simulated by an ensemble average of impulsive waves after shifting in time in such a way that the wave peaks are arranged in a line, irrespective of the travel time fluctuations caused by the longer wavelength component of the velocity fluctuation compared with the characteristic wavelength $\lambda$ of the impulsive wave studied. Then we evaluate the $Q^{-1}$ value from the gradient of the regression line for the plots of the maximum amplitude versus distance. We illustrate this averaging process for the high-frequency limit ($ak > 1$) in Fig. 2(a). The maximum amplitude of the newly averaged impulsive wave $\bar{\phi}$, shown at the bottom of Fig. 2(a), probably does not decrease so strongly as that of the usual mean wave $\bar{\phi}$ previously shown in Fig. 1(b) even in the case of $ak \gg 1$.

It is difficult to define mathematically the averaging process with arranging the wave peaks in a line. Therefore, instead of this averaging process, we introduce new wave $\psi$, in which the travel time fluctuation caused by the longer wavelength component of the fractional velocity fluctuation is corrected, and take an ensemble average of the travel time corrected waves $\psi$ by the usual averaging method, as illustrated in Fig. 2(b). The attenuation of the mean wave $\bar{\psi}$ is more adequate for the amplitude attenuation of impulsive waves in comparison with that of the usual mean wave formalism. We call this new statistical averaging method "a travel time corrected mean wave formalism." The longer wavelength component of the velocity structure

![Diagram](image)

FIG. 1. Schematic illustration of the averaging process for impulsive waves in the usual mean wave formalism (N experiments). The mean wave $\bar{\psi}$ is shown at the bottom. (a) Small attenuation in the low-frequency limit ($ak \ll 1$). (b) Large attenuation caused by large travel time fluctuations in the high-frequency limit ($ak \gg 1$).

![Diagram](image)

FIG. 2. Schematic illustration of averaging methods appropriate for the amplitude attenuation measurement of impulsive waves in the high-frequency limit ($ak \gg 1$) (N experiments). (a) Averaging waves $\psi$ after shifting in time so as to arrange the wave peaks in a line. The averaged wave $\bar{\psi}$ is shown at the bottom. (b) Averaging the travel time corrected waves $\psi'$ in usual means, where $\phi(x, t) = \psi(x, t + \delta t^2(x))$. The mean wave $\bar{\psi}'$ is shown at the bottom.
mainly causes the travel time fluctuation of propagating impulsive waves, and the shorter wavelength component of the velocity structure causes deformation or collapse of their shape by scattering.

II. TRAVEL TIME CORRECTED MEAN WAVE FORMALISM

The fractional velocity fluctuation $\xi(x)$ is written in the Fourier integral form:

$$\xi(x) = \frac{1}{(2\pi)^3} \int \int \int \xi(m) e^{i\mathbf{m}\cdot\mathbf{x}} dm,$$

where $\mathbf{m}$ is a wavenumber vector and $\xi(m)$ is the fractional velocity fluctuation in wavenumber vector space. We assume that the fractional velocity fluctuation with wavelength longer than twice the wavelength $\lambda$ studied causes travel time fluctuation $\delta t^c$, because such wavelengths can sample the fractional velocity fluctuation with an identical sign over the wavelength $\lambda$. The velocity with wavelengths shorter than $2\lambda$, we can easily show that the corresponding $Q^{-1}$ is larger than $Q^{-1}$ for the boundary wavelength of $2\lambda$. We decompose $\xi$ into the shorter wavelength component $\xi^S$ and the longer wavelength component $\xi^L$:

$$\xi(x) = \xi^S(x) + \xi^L(x),$$

$$\xi^S(x) = \frac{1}{(2\pi)^3} \int \int \int \xi(m) e^{i\mathbf{m}\cdot\mathbf{x}} dm,$$

$$\xi^L(x) = \frac{1}{(2\pi)^3} \int \int \int \xi(m) e^{i\mathbf{m}\cdot\mathbf{x}} \left[ 1 - H\left(m - \frac{\lambda}{2}\right) \right] dm,$$

where $k = 2\pi/\lambda$ is the wavenumber studied, $m = |\mathbf{m}|$, $k/2$ is the boundary wavenumber, and $H(m)$ is the step function defined by

$$H(m) = \begin{cases} 1, & \text{for } m > 0, \\ 0, & \text{for } m < 0. \end{cases}$$

An ensemble average of the product of two fractional velocity fluctuations in a wavenumber vector space can be written as

$$\langle \xi(l)\xi(m) \rangle = (2\pi)^3 \delta^3(1 + m) P(l),$$

where $\delta^3(m)$ is the delta function in three-dimensional space. We obtain the orthogonal relation between the shorter wavelength component $\xi^S$ and the longer wavelength component $\xi^L$ using (11):

$$\xi^S(x)\xi^L(x') = 0.$$  

According to the above decomposition, the autocorrelation function $R$ and the power spectral density function $P$ are decomposed into two components:

$$R(u) = R^S(u) + R^L(u),$$

$$R^S(u) = \xi^S(x + u)\xi^S(x),$$

$$R^L(u) = \xi^L(x + u)\xi^L(x),$$

and

$$P(m) = P^S(m) + P^L(m),$$

$$P^S(m) = \int \int \int R^S(u) e^{-i\mathbf{m}\cdot\mathbf{u}} du = P(m) H\left(m - \frac{k}{2}\right),$$

$$P^L(m) = \int \int \int R^L(u) e^{-i\mathbf{m}\cdot\mathbf{u}} du = P(m) \left[ 1 - H\left(m - \frac{k}{2}\right) \right],$$

where the superscripts $S$ and $L$ mean the shorter wavelength component and the longer wavelength component, respectively. We schematically illustrate these decompositions in Figs. 3(a)–(c).

The travel time fluctuation $\delta t^c$ caused by the longer wavelength component of the fractional velocity fluctuation $\xi^L$ is written as an integral along a ray path neglecting the terms of the second-order small quantity:

$$\delta t^c(x) = \frac{1}{C} \int_{\text{ray path}} \xi^L(x') ds(x'),$$

where $ds$ is an infinitesimal line element along the ray. Considering wave propagation along the $x_1$ axis, we can rewrite (19) in the differential form:

$$\delta t^c(x) = \frac{1}{C} \xi^L(x)$$

and

$$\delta t^c(x) = \frac{1}{C} \xi^L(x)$$

According to the argument in the preceding section, we introduce new wave $\psi$ in which travel time fluctuation caused by the longer wavelength component of the fractional velocity fluctuation is corrected:

$$\phi(x, t) = \psi(x, t + \delta t^c(x)).$$

Let us consider monochromatic wave $\psi_0(x)$ of angular frequency $\omega_0$:

$$\phi(x, t) = \psi_0(x) e^{-i\omega_0 t}.$$
The wave equation for the new wave \( \tilde{\psi}_o \) is obtained from (1) by using (20)—(22) as

\[
(\nabla^2 + k_o^2)\tilde{\psi}_o = 2k_0^2 \xi^2 \tilde{\psi}_o + 2k_0^2 \xi^2 \tilde{\psi}_o + i2k_0^2 \xi^2 \tilde{\psi}_o + i2k_0^2 \xi^2 \tilde{\psi}_o,
\]

where terms of the second-order small quantities are neglected. The first and the second terms represent the effect of the fractional velocity fluctuation. The third and the fourth terms show the correction of travel time fluctuation. We easily find that the second and the third terms cancel each other in the case of plane waves. The fourth term shows a fairly small effect of the slowly varying velocity gradient.

We solve (23) applying the binary interaction approximation in the mean wave formalism.\(^{13}\) We decompose the wave \( \tilde{\psi}_o \) into mean wave \( \tilde{\psi}_o \), coherent part, and fluctuating wave \( \tilde{\psi}_o' \) as

\[
\tilde{\psi}_o = \tilde{\psi}_o + \tilde{\psi}_o',
\]

where \( \tilde{\psi}_o = 0 \).

Assuming that the fluctuating wave \( \tilde{\psi}_o' \) is created from the first-order interaction between \( \tilde{\psi}_o \) and \( \xi^2 \) and \( \xi^4 \), we obtain the binary interaction approximation equation for the mean wave \( \tilde{\psi}_o \) by using the orthogonal relation (12):

\[
(\nabla^2 + k_o^2)\tilde{\psi}_o = \int \left[ 4k_0^2 R^2(u)G(u)\tilde{\psi}_o(x-u) + 4k_0^2 R^2(u)G(u)\tilde{\psi}_o(x-u) + 4ik_0^2 R^2(u)G(u)\tilde{\psi}_o(x-u) \\
+ 4ik_0^2 R^2(u)G(u)\tilde{\psi}_o(x-u) - 2k_0^2 R^2(u)\tilde{\psi}_o(x-u) - k_o^2 R^2(u)\tilde{\psi}_o(x-u) \\
- k_o^2 R^2(u)G(u)\tilde{\psi}_o(x-u) \right] du,
\]

where \( G \) is Green's function corresponding to outgoing waves, which satisfies

\[
(\nabla^2 + k_o^2)G(x) = \delta(x).
\]

Green's function can be written as a Fourier integral:

\[
G(x) = \frac{1}{(2\pi)^3} \int \int \int \tilde{G}(p)e^{ipx} dp.
\]

The integral kernel \( \tilde{G}(p) \) is given by

\[
\tilde{G}(p) = \text{Pr}[1/(k_o^2 - p^2)] - 2\pi \delta(k_o^2 - p^2),
\]

where \( p = |p| \), the symbol \( \text{Pr} \) means a principal integral kernel and \( \delta(m) \) is the delta function in one-dimensional space. We remember that the travel time fluctuation is corrected only for waves propagating along the \( x_1 \) axis. Solving (26) in the case of harmonic plane waves \( \tilde{\psi}_o(x) = e^{i\mathbf{p} \cdot \mathbf{x}}, \) where \( k = (k_0, 0, 0) \), we obtain the dispersion relation for the mean wave \( \tilde{\psi}_o \):

\[
k_o^2 - k^2 = \int \int \int \left[ 4k_0^2 R^2(u)G(u) + k_o^2(4k_o^2 R^2(u) - 4k_0 k + k^2) \\
- k_o^2 R^2(u) G(u) \right] e^{i\mathbf{p} \cdot \mathbf{x}} du.
\]

Terms on the right-hand side are the second-order small quantities. We can solve the dispersion relation (30) in perturbation, supposing that the deviation from the linear dispersion relation is small. Substituting \( k = k_o \), where \( k_o = (k_0, 0, 0) \), into the right-hand side of (30), we obtain

\[
k_o^2 - k^2 = \int \int \int \left[ 4k_0^2 R^2(u)G(u) + R^2(u)[k_o^2 G(u) \\
+ 2ik_o \xi G(u)] \right] e^{i\mathbf{p} \cdot \mathbf{x}} du.
\]

We can rewrite (31) as an integral in wavenumber vector space:

\[
k_o^2 - k^2 = \frac{4k_o^2}{(2\pi)^3} \int \int \int \left[ k_o^2 G(u) + R^2(u)[k_o^2 G(u) \\
+ 2ik_o \xi G(u)] \right] e^{i\mathbf{p} \cdot \mathbf{x}} d\mathbf{p},
\]

where \( p_i \) is the \( x_i \) component of the vector \( p \).

We can define the \( Q^{-1} \)-value for the mean wave \( \tilde{\psi}_o \) as

\[
Q^{-1}(k_o) = -(1/k_o^2) \text{Im}(k_o^2 - k^2),
\]

where the symbol \( \text{Im} \) means taking an imaginary part. Substituting (29) into (32) and taking an imaginary part, we obtain \( Q^{-1} \) as the integral over scattering angle \( \theta \):

\[
Q^{-1}(k_o) = \frac{1}{2\pi^2} \int \int \int \left( k_o^2 R^2(\mathbf{p}) + \mathbf{p} \cdot \mathbf{p} \right) e^{-i\mathbf{p} \cdot \mathbf{k} - i\mathbf{p} \cdot \mathbf{k}_o} d\mathbf{p} \\
+ \frac{k_o^2}{2\pi} \int \int \int \left( k_o^2 R^2(\mathbf{p}) + \mathbf{p} \cdot \mathbf{p} \right) e^{-i\mathbf{p} \cdot \mathbf{k}_o} d\mathbf{p} \\
+ \frac{k_o^2}{2\pi} \int \int \int \left( k_o^2 R^2(\mathbf{p}) + \mathbf{p} \cdot \mathbf{p} \right) e^{-i\mathbf{p} \cdot \mathbf{k}_o} d\mathbf{p},
\]

where \( p \) can be interpreted as the wavenumber vector of scattered waves, \( \theta \) is a scattering angle measured along the \( x_1 \) axis, and the critical angle \( \theta_c = 2\sin^{-1}(1/2) \). The delta function in (34) assures the energy conservation during the scattering. We schematically illustrate the wavenumber vector exchange during the scattering in Fig. 4. Comparing (34) with (5) obtained from usual statistical theories, we find out that the correction of travel time fluctuation gives the strong decreasing factor \( \sin^4(\theta/2) \) in the integral inside of the cone around the forward direction. The integral of the shorter wavelength component \( P^2 \) contributes mainly to the attenuation. In relation to this aspect, Chernov\(^{1} \) and Aki\(^{1} \) pointed out that the energy scattered around the forward direction did not contribute to the attenuation based on the Born approximation, and Wu\(^{12} \) calculated the \( Q^{-1} \) value of scalar waves only counting the energy scattered into the backward half-space.

Haruo Sato: Attenuation in random media

Earlier, the present authors derived $Q^{-1}$ for waves in one-dimensional elastic media with homogeneous random fluctuations of the square velocity $V^2_y(y) = V^2_0 + \delta V^2(y)$ and the mass density $\rho(y) = \rho_0 + \delta \rho(y)$ based on the usual mean wave formalism, where $y$ was a distance and the subscript 0 represented mean value. The one-dimensional power spectral density functions $P_x$ and $P_y$ were defined by
\[ P_x(m) = \int_{-\infty}^{\infty} \eta_x(y)m \eta_x(y) e^{-i\omega m} du, \]
where the fractional fluctuations $\eta_y$ and $\eta_x$ were defined by
\[ \eta_x(y) = \frac{5 \delta V^2(y)}{V^2_0} \quad \text{and} \quad \eta_y(y) = \frac{5 \delta \rho(y)}{\rho_0}. \]

The $Q^{-1}$ value for the mean wave was given by
\[ Q^{-1}(k_0) = (k_0/4)[P_x(0) + P_y(2k_0)]. \]

If we were to apply to the amplitude attenuation of impulsive waves, we should have neglected the forward scattering effect represented by the first term in (37) according to the preceding argument. We newly have $Q^{-1}$ only affected by the backward scattering based on a travel time corrected mean wave formalism:
\[ Q^{-1}(k_0) = (k_0/4)P_y(2k_0). \]

The mass density fluctuation contributes to the amplitude attenuation as the velocity fluctuation, and their mutual coherence is also important.

**III. THE $Q^{-1}$ VALUE FOR THE VON KARMAN AUTOCORRELATION FUNCTION**

We calculate the $Q^{-1}$ value for the von Karman autocorrelation function \( \alpha \) as a typical example for random fractional velocity fluctuation.

The von Karman autocorrelation function of order $\nu$ is given by
\[ R_x(\mu) = \frac{\epsilon^2}{2 \Gamma(\nu/2)} K_\nu(\mu) \Gamma(\nu/2), \quad \text{for} \ 0 < \nu < \frac{1}{3}, \]
where $\Gamma(\nu)$ is the gamma function and $K_\nu(\mu)$ is the Bessel function of the second kind of imaginary argument of order $\nu$. The exponential autocorrelation function is given by the case of order $1/2$. The corresponding power spectral density function is given by
\[ P_x(m) = \frac{\epsilon^2(2\pi^2 a^2)^3 \Gamma(\nu + 3/2)}{\Gamma(\nu)(1 + a^2 m^2)^{3+\nu/2}}, \]

Asymptotic behaviors are given by
\[ P_x(m) = \begin{cases} \frac{\epsilon^2(2\pi^2 a^2)^3 \Gamma(\nu + 3/2)}{\Gamma(\nu)}, \quad \text{for} \ am \ll 1, \\ \frac{\epsilon^2(2\pi^2 a^2)^3 \Gamma(\nu + 3/2)}{\Gamma(\nu)(am)^{3+\nu/2}}, \quad \text{for} \ am \gg 1. \end{cases} \]

In the high-frequency limit ($am \gg 1$), the power spectral density function decreases with the frequency of power $-2\nu - 3$. The smaller the order $\nu$, the more the short wavelength components are contained in the fractional velocity fluctuation. We illustrate the von Karman autocorrelation and the corresponding power spectral density functions in Figs. 5(a) and (b), respectively. We rewrite (34) in a more convenient form for integration.

![Graph](image-url)
\[ Q^{-1}(k_0) = \frac{\beta}{2\pi} \left( \int_0^1 z^2 P(2^{1/2}k_0z^{1/2})dz + \frac{1}{4} \int_0^{1/\alpha} z^2 P(2^{1/2}k_0z^{1/2})dz \right), \]  
where the latter corresponds to scattering around the forward direction.

Substituting (40) into (42), we obtain

\[ Q_\nu^{-1}(k_0) = \frac{\epsilon^2 2 \alpha \nu^2 \Gamma(1/2)}{\Gamma(1/4)} \left\{ \frac{1 + 4a^2k_0^2}{(1 + 4a^2k_0^2)} - 2 \frac{1}{(1 + 4a^2k_0^2)} \right\} 1 + \frac{1}{2} \epsilon^2 \ln(1 + a^2k_0^2), \]  
for \( 0 < \nu < \frac{1}{2}, \)  
and especially

\[ Q_1^{-1}(k_0) = \epsilon^2 2a k_0 \frac{1}{1 + a^2k_0^2/4} - \frac{1}{1 + 4a^2k_0^2} + \frac{\epsilon^2}{16a k_0} \frac{1}{1 + a^2k_0^2/4} - \frac{4}{a^2k_0^2} \ln(1 + a^2k_0^2/4), \]  
for \( \nu = \frac{1}{2}, \)  
where the subscript \( \nu \) represents the order of the von Karman autocorrelation function. In the limiting case, we obtain

\[ Q_\nu^{-1}(k_0) = \frac{\epsilon^2 \nu \Gamma(1/2)}{\Gamma(1/4)} \left\{ \frac{1}{1 + a^2k_0^2/4} - \frac{1}{1 + 4a^2k_0^2} \right\} + \frac{\epsilon^2}{16a k_0} \frac{1}{1 + a^2k_0^2/4} - \frac{4}{a^2k_0^2} \ln(1 + a^2k_0^2/4), \]  
for \( \nu = \frac{1}{2}. \)

We illustrate the plots of \( Q^{-1} \) versus \( ak_0 \) in Fig. 6. Each \( Q^{-1} \) has a peak of the order of \( \epsilon^2 \) around \( ak_0 = 2, \) the wavelength of \( \pi \) times as long as the correlation distance. Strictly speaking, the more the order \( \nu \) increases, the larger the peak value increases and the lower the peak frequency becomes. In the low-frequency limit \( (ak_0 \ll 1), \) every \( Q^{-1} \) is proportional to the third power of frequency, the Rayleigh scattering. In the high-frequency limit \( (ak_0 \gg 1), \) \( Q^{-1} \) decreases with frequency of negative power \(-2\nu\). The \( Q^{-1} \) value newly obtained does not increase with frequency in the high-frequency limit. Such a frequency dependence qualitatively matches well the conjectured amplitude attenuation of impulsive waves.

Thus introducing a new statistical averaging method appropriate for the amplitude attenuation measurement of impulsive waves, we have succeeded in deriving \( Q^{-1} \) applicable for a wide range of frequencies. As a more interesting problem, especially in seismology, it remains for us to construct an amplitude attenuation theory for impulsive vector waves in random elastic media, when not only the velocity fluctuation but also the mass density fluctuation contributes to the attenuation as seen in the one-dimensional case. The present author believes that the correction of the travel time fluctuation will prevent the \( Q^{-1} \) value from increasing with frequency in the high-frequency limit even in the case of vector waves.

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