Is the Single Scattering Model Invalid for the Coda Excitation at Long Lapse Time?

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Abstract—Supposing that the distribution of scatterers in a three-dimensional medium is not uniform but fractally homogeneous with fractal dimension $D$, we have made the dimensional analysis for the temporal decay of the multiple scattering energy density at the hypocenter. The number of scatterers in a sphere of radius $R$ is assumed to be proportional to $R^D$. Then, the energy density of the $k$th order scattering decays according to the $[(D-2)k-3]$th power of lapse time. A fractal dimension of $D=3$ corresponds to the uniform distribution. If $2 < D \leq 3$, multiple scattering terms of order $k \geq 2$ dominate over the single scattering term ($k = 1$) at long lapse time. If $D = 2$, energy density of every order decays according to the $-3$rd power of lapse time. The single scattering model survives on condition $D < 2$; the single scattering term dominates over the higher order multiple scattering terms even at long lapse time, since the negative power of lapse time for $k = 1$ is the smallest of all.

Key words: Coda, fractal, scattering, seismogram.

1. Introduction

On the basis of the homogeneous distribution of point-like isotropic scatterers in three-dimensional infinite space (cf. SATO, 1977) and the spherical radiation from the hypocenter, we can calculate the energy density for scattered waves at the hypocenter at lapse time $t$ as power series of scattering cross-section $\sigma$:

$$E^S(t) = W \cdot \left[ n\sigma / (2\pi V^2 t^2) + n(na)^2 / (16Vt) + \cdots C_k \cdot (n\sigma)^k \cdot (Vt)^{k-3} + \cdots \right]$$

(1)

where parameters are as follows: $W$, the radiated energy; $n$, the number density of scatterers being constant; $V$, wave velocity; $C_k$, the coefficient of the $k$th power of $\sigma$ (cf., KOPNIEV, 1977; GAO et al., 1983). Here, the geometrical spreading for the seismic wave energy was assumed to be proportional to the inverse square of distance. The first term originally obtained by AKI and CHOUET (1975) is called the single scattering model for the coda excitation. The power of lapse time is $k - 3$ for the $k$th order multiple scattering. As lapse time increases, it has been believed that

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multiple scattering terms of $k \geq 2$ dominate over the single scattering term ($k = 1$). But, is this solidly established? Here, in this paper, we will investigate what parameter determines the power of lapse time and the possibility that the single scattering model is valid at long lapse time.

2. Fractal Structure

We adopt that high frequency coda waves are mainly composed of scattered $S$ waves (Akı and Chouet, 1975). It means that the geometrical spreading of seismic wave energy is inversely proportional to the second power of distance. Here, let me doubt the existence of the volume independent number density of scatterers for seismic waves in the lithosphere. As is well-known in astronomy, the global density of matter, which is defined as the ratio of mass in a sphere of radius $R$ to its volume, does not converge to a finite limit. The global density decreases as the volume studied increases because of clustering of galaxies. If we write the mass to be proportional to $R^D$, then the global density is proportional to $R^{D-3}$, where the parameter $D$ can be interpreted as the fractal dimension (Mandelbrot, 1982, pp. 85 and 91; de Vaucouleurs, 1970). The recent study of the galaxy-galaxy correlation (Szalay and Schramm, 1985) showed a fractal dimension of $D = 1.2$ being much smaller than $3$. We know that the spatial distribution of earthquake epicenters have stochastic self similarities with $D = 1.0 \sim 1.6$ in two-dimensional space (Sadovskiy et al., 1984; Kagan and Knopoff, 1980). Furthermore, the distribution of acoustic emission hypocenters in a rock sample under a uniaxial loading has a fractal structure with $D = 2.25 \sim 2.75$ for granite (Hirata et al., 1987a) and $D = 1.47 \sim 2.0$ for andesite (Hirata et al., 1987b). There is a report on the decreasing $Q_c^{-1}$ with depth in France (Gagnepain-Beyneix, 1987). It does not necessarily mean the fractal structure, but it is related to the nonuniform distribution of scatterers. Thus, we have few observational bases for the constant number density of scatterers. Therefore, it is natural for us to imagine that the distribution of scatterers such as cracks, and/or various kinds of inclusions is not uniform but fractally homogeneous in the lithosphere.

Here, as the most simple introduction of fractal homogeneity, let us assume that the number density of scatterers at coordinate $x$ is not as constant but a function of distance $r = |x|$ from the origin as

$$n(r) = F \cdot r^{D-3}$$

(2)

where the dimension of coefficient $F$, $[F] = L^{-D}$. $L$ is the dimension of length. Then, the number of scatterers $N(R)$ in a sphere of radius $R$ is

$$N(R) = \int_{0}^{R} n(r) 4\pi r^2 \, dr = (4\pi F/D)R^D.$$  

(3)
The wave energy is supposed to be spherically radiated from the origin. Then, we have the single scattering energy density at the origin as

\[ E_s^c(t) = W\sigma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x \cdot [n(r)/(4\pi r^2)]\delta(Vt - 2r) \]

\[ = W\sigma F[(\pi/2)^{(D-2)} \cdot (Vt)^{(5-D)}] \propto W(\sigma F)^1 \cdot (Vt)^D - 5, \tag{4} \]

where the delta function means the scattering shell (SATO, 1977). The power of lapse time is \(-2\) for \(D = 3\), \(-3\) for \(D = 2\), and \(-4\) for \(D = 1\).

The double scattering energy density \((k = 2)\) at the origin is given by the volumetric integral for the first scatterer at \(x_1\) and the second scatterer at \(x_2\):

\[ E_s^2(t) = W\sigma^2 \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x_1 \cdot n(r_1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x_2 \cdot n(r_2) \]

\[ \cdot \left\{ \delta(Vt - r_1 - r_{1,2} - r_2) /[4(\pi r_1^2)(4\pi r_{1,2}^2)(4\pi r_2^2)] \right\} \]

\[ = [W\sigma^2/(4\pi Vt)] \cdot \int_{-1}^{1} dz \cdot [n(Vt^2z^2)/z] \cdot \int_{-1}^{1} dy \cdot \{n(Vt^2y^2)/[y(2 - z - y)]\} \]

\[ = \{W(\sigma F)^2/[\pi 4^{D-2}(Vt)^{(7-2D)}]\} \cdot \int_{-1}^{1} dz [z^D - 4] \cdot \int_{-1}^{1} dy [y^D - 4]/(2 - z - y) \tag{5} \]

where \(r_1 = |x_1|\) and \(r_2 = |x_2|\) and \(r_{1,2} = |x_1 - x_2|\). Then, we get

\[ E_s^2(t) \propto W \cdot (\sigma F)^2 \cdot (Vt)^{2D - 7}. \tag{6} \]

The power of lapse time is \(-1\) for \(D = 3\), \(-3\) for \(D = 2\), and \(-5\) for \(D = 1\).

In general, the energy density of the \(k\)th order scattering at the origin is written as the multiple integral over \(k\) scatterers:

\[ E_s^k(t) = W\sigma^k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x_1 \cdot n(r_1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x_2 \cdot n(r_2) \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3x_k \cdot n(r_k) \]

\[ \cdot \left\{ \delta(Vt - r_1 - r_{1,2} - \cdots - r_{k-1,k} - r_k) \right\} \]

\[ \cdot \{(4\pi)^{-k-1} \cdot r_1^{-2} \cdot r_{1,2}^{-2} \cdot r_{2,3}^{-2} \cdots r_{k-1,k}^{-2} \cdot r_k^{-2}\} \tag{7} \]

where \(r_1 = |x_1|\) and \(r_k = |x_k|\) and \(r_{i-1,i} = |x_{i-1} - x_i|\) for \(i = 2 \sim k\). On the basis of the dimensional analysis, we get

\[ E_s^k(t) \propto W \cdot (\sigma F)^k \cdot (Vt)^{(D-2)k - 3} \tag{8} \]

since \([\sigma] = L^2\). Its asymptotic behavior is essentially characterized by the power of lapse time, \((D - 2)k - 3\). The total scattering energy density at the origin is written as a sum:

\[ E_s^s(t) = \sum_{k=1}^{\infty} E_s^k(t). \tag{9} \]
If $2 < D \lesssim 3$, multiple scattering terms of order $k \geq 2$ dominate over the single scattering term ($k = 1$) at long lapse time. If $D = 2$, energy density of every order decays according to the $-3$rd power of lapse time. We note that the decay gradient of the single scattering energy density becomes the smallest of all on condition $D < 2$. That is, the single scattering term is dominant over the higher order multiple scattering terms at long lapse time.

We note that scattering energy loss was not considered here. It is the most simple way to multiply scattering attenuation $\exp(-Q^{-1} \cdot 2\pi ft)$ to (9) for frequency $\text{fHz}$; however, we cannot assure you of the energy conservation for the space integral of the sum of scattered wave energy density and the direct wave energy density. The energy conservation was proved only in the case of single scattering (Sato, 1977, Appendix B). We still have to solve this problem, but here we put it to one side.

Akı and Chouet (1975), assuming a constant number density for scatterers' distribution, interpreted that the power of lapse time for the geometrical factor depends on whether coda waves are composed of scattered surface waves or scattered body waves. Thus shown above; however, if we adopt that coda waves are scattered body waves, the power of lapse time for the geometrical factor is essentially determined by the fractal dimension $D$ of the scatterers' distribution in three-dimensional space. For example, Figure 9 of Akı and Chouet (1975) showed that the power of lapse time is $-3.48$ at 24 Hz in Kanto, Japan. Adopting (4), we may interpret that this power corresponds to $D = 1.52$. Then, the single scattering term is expected to dominate over multiple scattering terms of $k \geq 2$.

3. Discussion

We showed that the introduction of the fractally homogeneous distribution of scatterers modified the power of lapse time in the geometrical factor for the energy density of scattered waves. It will be left for us to examine the reality of the fractal structure of the scatterers' distribution in the lithosphere on the basis of the simultaneous determination of the power of lapse time in the geometrical factor and coda attenuation $Q^{-1} \cdot C$ from observed coda amplitude decay. It is interesting for us to study the regional differences and the frequency dependences of the power of lapse time. We have to formulate theoretically the scattering attenuation in fractal structure and also to examine observationally the change in scattering attenuation with travel distance. Decreasing of scattering attenuation with distance might be expected in the case of a small fractal dimension.

We have to note that the divergence $n(0)$ for $D < 3$ may prevent integrals (5) and (7) from convergence. But, we can overcome such a mathematical difficulty by introducing cut-off radius $a$ for the power law (2) as $n(r) = \text{constant for } r < a$. Various kinds of numerical simulation studies of scattering in fractal structure will be
necessary. For example, the study of coda excitation in the Fournier universe (Mandelbrot, 1982, p. 86) is very interesting.

Wu and Aki (1985) studied the fractal nature of the lithosphere as an inhomogeneous elastic medium from the spectral structure of coda excitation strength and scattering attenuation. It is necessary for us to investigate the relation between such a spectral structure and the fractal dimension of the point-like scatterers’ distribution here studied.

REFERENCES


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