Three-component seismogram envelope synthesis in randomly inhomogeneous semi-infinite media based on the single scattering approximation

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Abstract

We present a method for synthesizing three-component seismogram envelopes of local earthquake in a randomly inhomogeneous semi-infinite medium. The method extends the single scattering model of Sato [Sato, H., 1984. Attenuation and envelope formation of three-component seismograms of small local earthquakes in randomly inhomogeneous lithosphere. J. Geophys. Res. 89: 1221-1241.] by incorporating the effects of a free surface, frequency-dependent non-spherical radiation from a double-couple point source, and non-isotropic scattering including wave-type conversion. We synthesize seismogram envelopes—the root-mean-square (rms) amplitude of seismograms—at a receiver located on the free surface by dividing an inhomogeneous medium into many small cells and summing the energy of scattered waves from the cells on the isochronal scattering shells of different scattering modes for a given lapse time. Our main focus is the free surface effect, which alters the shape of isochronal scattering shells and the amplitudes of incident waves. Synthesizing three-component seismogram envelopes for different source-receiver configurations, we find that these effects on seismogram envelopes are not negligible and are pronounced at early S coda. P coda envelope shape is not very sensitive to free-surface incorporation because of dominant SP scattered waves which come from the source direction. © 1997 Elsevier Science B.V.

Keywords: Seismogram envelopes; S coda; P coda; Local earthquakes

1. Introduction

The study of local earthquake seismograms has been used extensively in recent years to understand the random inhomogeneity of the earth's crust and lithosphere. However, most forward modeling of seismic scattering has concentrated on constructing single-component seismogram envelopes rather than envelopes for three-components of motion (e.g., Aki and Chouet, 1975; Sato, 1977, 1989, 1995; Wu, 1985; Gusev and

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Abubakirov, 1987; Hoshiba, 1991; Zeng et al., 1991). These studies were largely devoted to the synthesis of S coda because of the mathematical ease of calculation. In spite of the analytic efficiency of observed seismogram envelopes, a purely shear-wave computation is inadequate for modeling envelopes that differ strongly for different components of motion (e.g., at short lapse times, especially near the arrival of direct waves). To obtain more precise information about the inhomogeneous structure of the earth media from observed seismogram envelopes, we have to construct a synthetic method for more realistic three-component envelopes.

Mathematical frameworks for the synthesis of three-component seismogram envelopes (or their original seismograms) can be classified into deterministic and stochastic approaches. These methods differ in how to describe or model the inhomogeneous property of the earth. Wang and Herrmann (1988) studied the features of three-component seismograms which were generated by point elastic inclusions in a layered medium. By using simple inhomogeneous structure models, in which spherical inclusions were deterministically embedded in a homogeneous elastic medium, Craig et al. (1991) synthesized three-component seismograms on the basis of the single scattering approximation. Though these deterministic approaches allow a simple interpretation of the relation between inhomogeneous structures and seismogram envelopes, it is uncertain if spherical or point-like inclusions are adequate to represent earth inhomogeneity. Random inhomogeneities of subsurface structures reported from well logging measurements (e.g., Wu et al., 1994; Shiomi et al., 1996) support a stochastic rather than a deterministic approach. Sato (1984) proposed a stochastic method of synthesizing three-component seismogram envelopes for local earthquakes. He evaluated the first order or Born scattering amplitude of elastic waves, then constructed seismogram envelopes by taking an incoherent sum of the single-scattered seismic waves. Synthetic envelopes obtained in his study incorporated the effects of the non-spherical source radiation and the non-isotropic scattering, but the effect of the free surface was ignored.

In this study, we extend the mathematical framework proposed by Sato (1984) to randomly inhomogeneous semi-infinite media in order to construct more realistic three-component seismogram envelopes as recorded on the free surface. Based on the method proposed here, we synthesize three-component seismogram envelopes for various source–receiver configurations and discuss how the presence of the free surface alters the shape of seismogram envelopes. We also investigate the effect of the source radiation pattern on envelope shapes in relation with the dominant scattering mode at certain lapse times.

2. Synthetic method

In this section, we describe the characteristics of our randomly inhomogeneous elastic media and our earthquake source model. These descriptions enable us to derive simple synthetic formulas for the three-component seismogram envelopes in randomly inhomogeneous infinite media. Then, synthetic formulas in a prolate spheroidal coordinate system given by Sato (1984) are rederived in Cartesian coordinates more appropriate for a medium with a free surface.

2.1. Characterization of randomly inhomogeneous elastic media

We assume a randomly inhomogeneous elastic medium, in which the inhomogeneity is spatially uniform and isotropic. Let us describe the spatial variation of locally isotropic elastic parameters as a function of position in the Cartesian coordinate $x = (x, y, z)$:

$$
\begin{align*}
\rho(x) &= \rho_0 + \delta \rho(x) \\
\alpha(x) &= \alpha_0 + \delta \alpha(x) \\
\beta(x) &= \beta_0 + \delta \beta(x)
\end{align*}
$$

(1)
where $\rho$ is the density, and $\alpha$ and $\beta$ are the P and S wave velocities. The subscript zero and symbol $\delta$ represent the mean value and fluctuation of the elastic parameters. We assume that the fractional fluctuations of the elastic parameters are sufficiently small compared with the mean values:

$$|\delta \rho / \rho_0|, |\delta \alpha / \alpha_0|, |\delta \beta / \beta_0| \ll 1$$

(2)

For simplicity, we assume a similarity among the fractional fluctuations of each elastic parameter:

$$\xi(x) = \nu^{-1} \frac{\delta \rho(x)}{\rho_0} = \frac{\delta \alpha(x)}{\alpha_0} = \frac{\delta \beta(x)}{\beta_0}$$

(3)

where $\nu$ is a proportionality factor. The equality between the fractional fluctuations of the density and the elastic wave velocities in Eq. (3) is supported by Birch's empirical law based on various rock experiments (Birch, 1961), and we take $\nu = 0.8$ (Sato, 1984). We now imagine an ensemble of random function $\xi(x)$. We take $\xi$ as

$$\langle \xi(x) \rangle = 0$$

(4)

where $\langle \rangle$ represents the ensemble average.

To describe the stochastic characteristics of inhomogeneity, we define the auto-correlation function of $\xi$ as

$$R_\xi(w) = \langle \xi(x + w) \xi(x) \rangle$$

(5)

where the argument is lag $w = |w|$ and independent of $x$. Then the power spectral density function of $\xi$ is given by

$$\Xi_\xi(m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_\xi(w) e^{-i(mw)} dw$$

(6)

where the argument is wavenumber $m = |m|$. In this study, we assume an auto-correlation function of exponential type:

$$R_\xi(w) = \sigma^2 e^{-w^2/a}$$

(7)

Thus the random media are statistically characterized by only two parameters: the correlation length $a$ and the rms fractional fluctuation strength $\sigma$.

Fig. 1. Configuration of the earthquake source, the scatterer and the receiver in an infinite medium. Ray paths are schematically shown by thin lines. Each local coordinate system is defined to be a right-hand system (see text for details).
2.2. Description of earthquake source

We put a double-couple point source at the origin of the Cartesian coordinate system (Fig. 1). Let us denote strike, dip, and rake of the double-couple point source by $\phi_s$, $\delta$, and $\lambda$. We describe a set of the orthogonal unit vectors for the Cartesian coordinate system as $(\hat{x}, \hat{y}, \hat{z})$, where these vectors point North, East, and vertically downward, respectively. For a homogeneous infinite medium, the source excites a far-field displacement $u$ at receiver $x_r = (x_r, y_r, z_r)$ given by Aki and Richards (1980) p. 115 as

$$u(x_r, t) = B_p \hat{e}_{er1} \frac{2M(t - \frac{r_{er}}{c})}{\sqrt{15 \cdot 4\pi \rho_0 \alpha_0 r_{er}}} + [B_{SV} \hat{e}_{er2} + B_{SH} \hat{e}_{er3}] \sqrt{\frac{2}{5}} \frac{M(t - \frac{r_{er}}{c})}{4\pi \rho_0 \beta_0^3 r_{er}}$$

with

$$\hat{e}_{er1} = (x_r / r_{er}, y_r / r_{er}, z_r / r_{er})$$
$$\hat{e}_{er2} = \hat{e}_{er3} \times \hat{e}_{er1}$$
$$\hat{e}_{er3} = \left\{ \begin{array}{ll}
\hat{z} \times \hat{e}_{er1} & \text{for } |\hat{z} \times \hat{e}_{er1}| \neq 0 \\
\hat{y} & \text{for } |\hat{z} \times \hat{e}_{er1}| = 0
\end{array} \right. \tag{9}$$

and

$$B_p = \sqrt{\frac{15}{2}} \left[ \cos \lambda \sin \delta \sin^2 i_\xi \sin 2(\phi - \phi_s) - \cos \lambda \cos \delta \sin 2 i_\xi \cos(\phi - \phi_s) \ight.\left. + \sin \lambda \sin 2 \delta \left( \cos^2 i_\xi - \sin^2 i_\xi \sin^2(\phi - \phi_s) \right) + \sin \lambda \cos 2 \delta \sin 2 i_\xi \sin(\phi - \phi_s) \right] \tag{10}$$

$$B_{SV} = \sqrt{\frac{5}{2}} \left[ \sin \lambda \cos 2 \delta \cos 2 i_\xi \sin(\phi - \phi_s) - \cos \lambda \cos \delta \cos 2 i_\xi \cos(\phi - \phi_s) \ight.\left. + \frac{1}{2} \cos \lambda \sin \delta \sin 2 i_\xi \sin 2(\phi - \phi_s) - \frac{1}{2} \sin \lambda \sin 2 \delta \sin 2 i_\xi \left( 1 + \sin^2(\phi - \phi_s) \right) \right] \tag{11}$$

$$B_{SH} = \sqrt{\frac{5}{2}} \left[ \cos \lambda \cos \delta \cos i_\xi \sin(\phi - \phi_s) + \cos \lambda \sin \delta \sin i_\xi \cos 2(\phi - \phi_s) \ight.\left. + \sin \lambda \cos 2 \delta \cos i_\xi \cos(\phi - \phi_s) - \frac{1}{2} \sin \lambda \sin 2 \delta \sin i_\xi \sin 2(\phi - \phi_s) \right] \tag{12}$$

In Eq. (8), $t$ is time with the earthquake occurring at $t = 0$. $M(t)$ is the seismic moment function and the dot denotes its time derivative. The symbol $r_{er}$ is the distance between the earthquake source and the receiver. The set of the vectors $(\hat{e}_{er1}, \hat{e}_{er2}, \hat{e}_{er3})$ constitutes a local right-hand coordinate system. The terms $B_p$, $B_{SV}$, and $B_{SH}$ are the angular dependencies for P, SV and SH wave amplitudes, respectively, and are normalized as

$$\int B_{\phi}^2 d\Omega = \int \left[ B_{SV}^2 + B_{SH}^2 \right] d\Omega = 4\pi \tag{13}$$

for the solid angle integration over the polar take-off angle $i_\xi$ and azimuthal angle $\phi$. 
The energy spectral density for the radiated P and S waves may be written as

$$W_p(\omega) = \frac{\omega^4 |\tilde{M}(\omega)|^2}{15\pi \rho_0 c_0^5}$$
$$W_s(\omega) = \frac{\omega^4 |\tilde{M}(\omega)|^2}{10\pi \rho_0 c_0^5}$$

where $\omega$ is the angular frequency and $\tilde{M}(\omega)$ is the Fourier transform of the seismic moment function. These energy spectral densities are defined for the sum of the kinetic and potential energy.

For the description of the earthquake source, we adopt a $\omega$-square source spectral model:

$$\tilde{M}(\omega) = \frac{i}{\omega} \frac{M_0}{1 + (\omega/\omega_c)^2}$$

where $i = \sqrt{-1}$, $\omega_c$ is the angular corner frequency, and $M_0$ is the static seismic moment. Following the empirical relation of Thatcher and Hanks (1973), the static seismic moment of a local earthquake is related to its local magnitude $M_L$ by

$$\log M_0 = 1.5 M_L + 9.0$$

The other parameter of Eq. (15), the angular corner frequency, can be obtained through the empirical relation (Watanabe, 1971):

$$\log \omega_c = 1.5 + \log 2\pi - 0.20 M_L$$

By using these equations, we assign the source moment spectrum $\tilde{M}(\omega)$ solely by the local earthquake magnitude.

2.3. Seismogram envelopes in inhomogeneous infinite media

Throughout this study, we assume that the time history of source radiated energy, $S$, is a constant over the source duration time $T_0$:

$$S(T_0, t) = \begin{cases} 1 & 0 < t < T_0 \\ 0 & \text{otherwise} \end{cases}$$

In this case, we get the energy flux density of seismic waves at angular frequency $\omega$ at the receiver as

$$S \left( \frac{T_0 \cdot t}{\alpha_0} - \frac{r_{ct}}{\alpha_0} \right) \frac{W_p(\omega)}{4\pi r_{ct}^2} B_p^2 + S \left( \frac{T_0 \cdot t}{\beta_0} - \frac{r_{ct}}{\beta_0} \right) \frac{W_s(\omega)}{4\pi r_{ct}^2} \left[ B_{sv}^2 + B_{sh}^2 \right]$$

where $(4\pi r_{ct}^2)^{-1}$ represents geometrical spreading. This expression is valid for the homogeneous infinite medium.

Consider now the power spectral density of the velocity amplitude of seismic waves at the receiver in an inhomogeneous infinite medium. We define the power spectral density $P_i(x_r, t | \omega)$ in the time interval $\Delta T$ as

$$P_i(x_r, t | \omega) = \frac{1}{\Delta T} \left| \int_0^{\Delta T} \hat{u}_i(x_r, t + t') e^{i\omega t'} dt' \right|^2$$
where the subscript $i$ denotes the Cartesian component of motion. Taking $\Delta T > T_0$, for scattered waves, we assume that the velocity amplitude $\hat{u}_i$ can be regarded as quasi-stationary within the time interval. For the estimation of the power spectral density for direct waves, we may choose $\Delta T$ as small as source duration $T_0$.

Since small travel time fluctuations are irrelevant to our main subject, we assume constant P and S wave propagation velocities in the inhomogeneous medium. In this case, the power spectral densities for direct P and S waves at the receiver can be written as

$$P_i^P(x_r, t|\omega) = \frac{1}{\rho_0 \alpha_0} S \left( \frac{T_0}{T} t - \frac{r_{et}}{\alpha_0} \right) W_p(\omega) \left\| A G^P \right\|^2$$

$$P_i^S(x_r, t|\omega) = \frac{1}{\rho_0 \beta_0} S \left( \frac{T_0}{T} t - \frac{r_{et}}{\beta_0} \right) W_s(\omega) \left\| A G^S \right\|^2$$

with

$$E^P = \begin{pmatrix} B_P \\ 0 \\ 0 \end{pmatrix}$$

$$E^S = \begin{pmatrix} 0 \\ B_{SV} \\ B_{SH} \end{pmatrix}$$

$$G^P_{et} = \begin{pmatrix} 1 \\ \sqrt{4\pi} r_{et} \exp \left( - \frac{r_{et} \omega Q_p^{-1}(\omega)}{2\alpha_0} \right) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$G^S_{et} = \begin{pmatrix} 0 \\ \sqrt{4\pi} r_{et} \exp \left( - \frac{r_{et} \omega Q_s^{-1}(\omega)}{2\beta_0} \right) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \hat{e}_{er1} \cdot \hat{x} \\ \hat{e}_{er2} \cdot \hat{x} \\ \hat{e}_{er3} \cdot \hat{x} \\ \hat{e}_{er1} \cdot \hat{y} \\ \hat{e}_{er2} \cdot \hat{y} \\ \hat{e}_{er3} \cdot \hat{y} \\ \hat{e}_{er1} \cdot \hat{z} \\ \hat{e}_{er2} \cdot \hat{z} \\ \hat{e}_{er3} \cdot \hat{z} \end{pmatrix}$$

The notation $\| n^2$ signifies the square of the $i$-th component of the vector which is given by matrix product. $E$, $G_{et}$ and $A$ are matrix notation for the source, propagation and coordinate terms, respectively. Since we consider seismic wave propagation through an inhomogeneous medium, we introduce terms such as $\exp(- r_{et} \omega Q_p^{-1}(\omega)/2\alpha_0)$ in the matrix $G_{et}$ to account for scattering attenuation (see Appendix B). For practical calculations, the effect of intrinsic attenuation, not considered here, can be taken into account in the same manner.
On the basis of the single scattering approximation, we next calculate the power spectral densities of scattered waves. For simplicity, let us first consider P to S scattered waves only. Hereafter we adopt the notation that, for example, PS denotes P to S scattering. Suppose that a direct P wave at the point \( \mathbf{x}_s = (x_s, y_s, z_s) \) is scattered by a small inhomogeneous cell of finite volume \( L^3 \). We call this cell the scatterer. Such PS scattered waves observed at the receiver come from scatterers distributed throughout the infinite medium. Let us assume these arrivals are incoherent. In the form of an ensemble average over many realizations of spatial inhomogeneity, the power spectral density for PS scattered waves is written as

\[
\langle P^P_S(\mathbf{x}_s, t| \omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_{0,t} - \frac{r_{es}}{\alpha_0} - \frac{r_{es}}{\beta_0} \right) W_P(\omega) \left| A_{es} G_{es}^S S^{PS} G_{es}^P E^P \right|^2,
\]  

(28)

with

\[
S^{PS} = \begin{pmatrix}
\sqrt{\left( \frac{\langle P^{PS}_1 \rangle}{L^3} \right)} & 0 & 0 \\
\sqrt{\left( \frac{\langle P^{PS}_2 \rangle}{L^3} \right)} & 0 & 0 \\
\sqrt{\left( \frac{\langle P^{PS}_3 \rangle}{L^3} \right)} & 0 & 0
\end{pmatrix}
\]

(29)

\[
G_{es}^P = \begin{pmatrix}
\frac{1}{\sqrt{4\pi r_{es}}} \exp \left( -\frac{r_{es} \omega Q_{P}^{-1}(\omega)}{2\alpha_0} \right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(30)

\[
G_{es}^S = \begin{pmatrix}
\frac{1}{r_{es}} \exp \left( -\frac{r_{es} \omega Q_{S}^{-1}(\omega)}{2\beta_0} \right) & 0 & 0 \\
0 & \frac{1}{r_{es}} \exp \left( -\frac{r_{es} \omega Q_{S}^{-1}(\omega)}{2\beta_0} \right) & 0 \\
0 & 0 & \frac{1}{r_{es}} \exp \left( -\frac{r_{es} \omega Q_{S}^{-1}(\omega)}{2\beta_0} \right)
\end{pmatrix}
\]

(31)

\[
A_{es} = \begin{pmatrix}
\hat{\mathbf{e}}_{es1} \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_{es2} \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_{es3} \cdot \hat{\mathbf{x}} \\
\hat{\mathbf{e}}_{es1} \cdot \hat{\mathbf{y}} & \hat{\mathbf{e}}_{es2} \cdot \hat{\mathbf{y}} & \hat{\mathbf{e}}_{es3} \cdot \hat{\mathbf{y}} \\
\hat{\mathbf{e}}_{es1} \cdot \hat{\mathbf{z}} & \hat{\mathbf{e}}_{es2} \cdot \hat{\mathbf{z}} & \hat{\mathbf{e}}_{es3} \cdot \hat{\mathbf{z}}
\end{pmatrix}
\]

(32)

\[
\hat{\mathbf{e}}_{es1} = \left( \frac{x_s}{r_{es}}, \frac{y_s}{r_{es}}, \frac{z_s}{r_{es}} \right)
\]

\[
\hat{\mathbf{e}}_{es2} = \hat{\mathbf{e}}_{es3} \times \hat{\mathbf{e}}_{es1}
\]

\[
\hat{\mathbf{e}}_{es3} = \begin{pmatrix}
\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1} / |\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1}| & \hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1} \\
\hat{\mathbf{y}} & \hat{\mathbf{y}}
\end{pmatrix} \text{ for } |\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1}| \neq 0
\]

(33)

\[
\hat{\mathbf{e}}_{es3} = \begin{pmatrix}
\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1} / |\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1}| & \hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1} \\
\hat{\mathbf{y}} & \hat{\mathbf{y}}
\end{pmatrix} \text{ for } |\hat{\mathbf{z}} \times \hat{\mathbf{e}}_{es1}| = 0
\]
where $r_{es}$ is the distance between the earthquake source and the scatterer, and $r_{st}$ is the distance between the scatterer and the receiver. The summation $\Sigma$ is taken over all scatterers in the infinite medium. The symbols $\langle |P_{i}^{PS}|^{2} \rangle/L^{3}$ denote mean-square amplitudes of PS scattered waves due to a single inhomogeneous cell of unit volume, and are written as a function of the power spectral density of inhomogeneities (see Appendix A). The matrix operator $A_{t}$ is used for the transformation of coordinates between ($\hat{e}_{es1}$, $\hat{e}_{es2}$, $\hat{e}_{es3}$) and ($\hat{x}$, $\hat{y}$, $\hat{z}$).

In the same way, the power spectral densities for scattered PP, SP and SS waves can be written as

$$
\langle P_{i}^{PP}(x_{t}, t | \omega) \rangle = \frac{1}{\rho_{0} \alpha_{0}} \sum S \left[ T_{0}, t - \frac{r_{es}}{\alpha_{0}} - \frac{r_{st}}{\alpha_{0}} \right] |W_{p}(\omega)|^{2} \left| A_{t} G_{st}^{PP} G_{st}^{PP} E_{t}^{P} \right|^{2}
$$

(34)

$$
\langle P_{i}^{SP}(x_{t}, t | \omega) \rangle = \frac{1}{\rho_{0} \beta_{0}} \sum S \left[ T_{0}, t - \frac{r_{es}}{\beta_{0}} - \frac{r_{st}}{\beta_{0}} \right] |W_{s}(\omega)|^{2} \left| A_{t} G_{st}^{SP} G_{st}^{SP} E_{t}^{S} \right|^{2}
$$

(35)

$$
\langle P_{i}^{SS}(x_{t}, t | \omega) \rangle = \frac{1}{\rho_{0} \beta_{0}} \sum S \left[ T_{0}, t - \frac{r_{es}}{\beta_{0}} - \frac{r_{st}}{\beta_{0}} \right] |W_{s}(\omega)|^{2} \left| A_{t} G_{st}^{SS} G_{st}^{SS} E_{t}^{S} \right|^{2}
$$

(36)

with

$$
G_{es}^{SS} = \begin{pmatrix}
\frac{1}{\sqrt{4 \pi}} \exp \left( - \frac{r_{es} \omega Q_{s}^{-1}(\omega)}{2 \beta_{0}} \right) & 0 & 0 \\
0 & \frac{1}{\sqrt{4 \pi}} \exp \left( - \frac{r_{es} \omega Q_{s}^{-1}(\omega)}{2 \beta_{0}} \right) & 0 \\
0 & 0 & \frac{1}{\sqrt{4 \pi}} \exp \left( - \frac{r_{es} \omega Q_{s}^{-1}(\omega)}{2 \beta_{0}} \right)
\end{pmatrix}
$$

(37)

$$
G_{st}^{PP} = \begin{pmatrix}
\frac{1}{r_{st}} \exp \left( - \frac{r_{st} \omega Q_{p}^{-1}(\omega)}{2 \alpha_{0}} \right) & 0 & 0 \\
0 & \frac{1}{r_{st}} \exp \left( - \frac{r_{st} \omega Q_{p}^{-1}(\omega)}{2 \alpha_{0}} \right) & 0 \\
0 & 0 & \frac{1}{r_{st}} \exp \left( - \frac{r_{st} \omega Q_{p}^{-1}(\omega)}{2 \alpha_{0}} \right)
\end{pmatrix}
$$

(38)

The matrix elements for the mean-square amplitudes of scattered waves that appear in these equations are summarized in Appendix C.

From the formulas presented above, the power spectral densities for all the seismic waves at the receiver $x_{t}$ is

$$
\langle P_{i}(x_{t}, t | \omega) \rangle = \langle P_{i}^{PP}(x_{t}, t | \omega) \rangle + \langle P_{i}^{SP}(x_{t}, t | \omega) \rangle + \langle P_{i}^{SS}(x_{t}, t | \omega) \rangle + \langle P_{i}^{ew}(x_{t}, t | \omega) \rangle
$$

(39)
where

$$\langle P^\text{sc}(x,t|\omega) \rangle = \langle P^\text{PP}(x,t|\omega) \rangle + \langle P^\text{PS}(x,t|\omega) \rangle + \langle P^\text{SP}(x,t|\omega) \rangle + \langle P^\text{SS}(x,t|\omega) \rangle$$

(40)

Using Parseval’s theorem, we express the rms velocity amplitude in inhomogeneous infinite media as

$$\sqrt{\frac{1}{\pi} \int_0^\infty \langle P_i(x,t|\omega) \rangle d\omega}$$

(41)

Hereafter, we call the rms velocity amplitude trace a seismogram envelope. These expressions for the three-component seismogram envelopes in inhomogeneous infinite media are identical with those derived by Sato (1984), however, our expressions written in Cartesian coordinates are more adequate to incorporate the free surface boundary affecting real seismograms in inhomogeneous semi-infinite media.

2.4. Seismogram envelopes in inhomogeneous semi-infinite media

As depicted in Fig. 2a, we put a receiver on the free surface. Envelope synthesis in infinite media uses the ray path represented by $l$. As a consequence of the reflection of the free surface, the additional ray path shown

Fig. 2. (a) Configuration of the earthquake source, the scatterer and the receiver in an inhomogeneous semi-infinite medium. (b) Vertical plane, $\nu$-plane 1, including the earthquake source and the scatterer. (c) Vertical plane, $\nu$-plane 2, including the scatterer and the receiver.
by \( I' \) is incorporated in a semi-infinite media model. Both ray paths are in the vertical plane which includes the earthquake source and the scatterer.

Let us denote the incidence and reflection angle of the seismic waves at the free surface as \( \theta_{es1} \) and \( \theta_{es2} \), respectively (Fig. 2b). The symbol \( r_{es1} \) is the distance from the source to the reflection point and \( r_{es2} \) is that from the reflection point to the scatterer. We denote the velocities of the seismic waves propagating over these distances as \( v_{e1} \) and \( v_{e2} \), respectively. For the conversion reflection at the free surface, \( v_{e1} \) and \( v_{e2} \) take different values: e.g., \( v_{e1} = \alpha_0 \) and \( v_{e2} = \beta_0 \) for P to S conversion reflection. The parameters introduced above can be estimated on the basis of the ray theory if we specify the configuration of the source and the scatterer. The amplitudes of source radiated seismic waves are represented in the form of P–SV and SH mode by Eq. (8). Thus, we can investigate the property of the wave propagation along \( I' \). The geometrical spreading factor for the ray path \( I' \) given by Cerveny and Ravindra (1971) is

\[
D = \left( \sqrt{4\pi} \left[ \left( \frac{r_{es1}}{v_{e1}} + \frac{r_{es2}}{v_{e2}} \right) \frac{r_{es1}}{\cos \theta_{es1}} + \frac{r_{es2}}{\cos \theta_{es2}} \right] \cos \theta_{es1} \right)^{-1}
\]

From these expressions, for the ray path \( I' \), we obtain the matrices corresponding to \( G_{es} \) for \( I \) as

\[
G_{es}^{PP} = \begin{pmatrix}
R_{pp} D_{pp} \exp \left( -\frac{(r_{es1} + r_{es2})}{2 \alpha_0} \omega Q_p^{-1}(\omega) \right) & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
G_{es}^{PS} = \begin{pmatrix}
R_{ps} D_{ps} \exp \left( -\frac{r_{es1}}{2 \alpha_0} \omega Q_p^{-1}(\omega) - \frac{r_{es2}}{2 \beta_0} \omega Q_s^{-1}(\omega) \right) & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
G_{es}^{SP} = \begin{pmatrix}
0 & R_{sp} D_{sp} \exp \left( -\frac{r_{es1}}{2 \beta_0} \omega Q_s^{-1}(\omega) - \frac{r_{es2}}{2 \alpha_0} \omega Q_s^{-1}(\omega) \right) \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
G_{es}^{SS} = \begin{pmatrix}
0 & 0 \\
0 & R_{ss(P-SV)} D_{ss} \exp \left( -\frac{(r_{es1} + r_{es2})}{2 \beta_0} \omega Q_s^{-1}(\omega) \right) \\
0 & 0 & R_{ss(SH)} D_{ss} \exp \left( -\frac{(r_{es1} + r_{es2})}{2 \beta_0} \omega Q_s^{-1}(\omega) \right)
\end{pmatrix}
\]

where the superscript of \( G_{es} \) and the subscript of \( D \) represents the conversion mode at the free surface: e.g., P
to S reflection conversion is denoted by PS. \( R \) is the reflection coefficient at the free surface (see Appendix D), where the first and second subscripts represent the conversion mode at the free surface and the type of reflection (e.g., S, H), respectively. For simplicity, we ignore inhomogeneous waves propagating along the free surface. Since the set of the orthogonal unit vectors \( (\hat{\text{e}}_{s1}, \hat{\text{e}}_{s2}, \hat{\text{e}}_{s3}) \) depicted in Fig. 2b is different from that for the ray path \( l \), we need to designate a new \( \hat{\text{e}}_{s1} \) to be parallel to the ray path \( l' \). In the following expressions, we do not explicitly denote the difference. In order to describe the amplitude of the seismic waves propagating from the reflection point to the scatterer, we introduce the orthogonal unit vectors \( (\hat{\text{e}}_{s1}, \hat{\text{e}}_{s2}, \hat{\text{e}}_{s3}) \). We define \( \hat{\text{e}}_{s1}' \) to be parallel to \( l' \). Then \( \hat{\text{e}}_{s2}' \) and \( \hat{\text{e}}_{s3}' \) are obtained by exchanging \( \hat{\text{e}}_{s1} \) in Eq. (33) with \( \hat{\text{e}}_{s1}' \).

To evaluate the free surface amplification at a receiver as the P–SV or SH problem (Fig. 2c), we define a set of the orthogonal unit vectors \( (\hat{\text{e}}_{sr1}, \hat{\text{e}}_{sr2}, \hat{\text{e}}_{sr3}) \) as

\[
\hat{\text{e}}_{sr1} = \left( \frac{x_r - x_s}{r_s}, \frac{y_r - y_s}{r_s}, \frac{z_r - z_s}{r_s} \right)
\]

\[
\hat{\text{e}}_{sr2} = \hat{\text{e}}_{sr3} \times \hat{\text{e}}_{sr1}
\]

\[
\hat{\text{e}}_{sr3} = \begin{cases} \hat{\text{z}} \times \hat{\text{e}}_{sr1} & \text{for} |\hat{\text{z}} \times \hat{\text{e}}_{sr1}| \neq 0 \\ \hat{\text{y}} & \text{for} |\hat{\text{z}} \times \hat{\text{e}}_{sr1}| = 0 \end{cases}
\]

Both \( \hat{\text{e}}_{sr1} \) and \( \hat{\text{e}}_{sr2} \) are on the vertical plane which includes the scatterer and the receiver. By using the representations above, we obtain the power spectral densities of scattered waves for 12 propagation modes as

\[
\langle P_i^{PP}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{P} S_{PP} G_{sr}^{P} E^P \right| \right|^2
\]

\[
\langle P_i^{PS}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{S} S_{PS} G_{sr}^{P} E^P \right| \right|^2
\]

\[
\langle P_i^{SP}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_s(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{S} S_{SP} G_{sr}^{S} E^S \right| \right|^2
\]

\[
\langle P_i^{SS}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_s(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{S} S_{SS} G_{sr}^{S} E^S \right| \right|^2
\]

\[
\langle P_i^{PPP}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es1}}{\alpha_0} - \frac{r_{es2}}{\alpha_0} - \frac{r_{sr}}{\alpha_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{P} S_{PPP} G_{sr}^{P} E^P \right| \right|^2
\]

\[
\langle P_i^{PPS}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es1}}{\alpha_0} - \frac{r_{es2}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{P} S_{PPS} G_{sr}^{P} E^P \right| \right|^2
\]

\[
\langle P_i^{SPS}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es1}}{\alpha_0} - \frac{r_{es2}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{P} S_{SPS} G_{sr}^{S} E^P \right| \right|^2
\]

\[
\langle P_i^{SSS}(x_r, t|\omega) \rangle = \frac{1}{\rho_0 \alpha_0} \sum S \left( T_0,t - \frac{r_{es1}}{\alpha_0} - \frac{r_{es2}}{\alpha_0} - \frac{r_{sr}}{\beta_0} \right) W_p(\omega) \left| \left| \mathbf{RA}_i G_{sr}^{P} S_{SSS} G_{sr}^{S} E^P \right| \right|^2
\]
\[
\langle P_{i}^{SP}(x_{r},t|\omega) \rangle = \frac{1}{\rho_{0}} \sum_{S} \left( T_{\text{source}}, t - \frac{r_{\text{es1}}}{\beta_{0}} + \frac{r_{\text{es2}}}{\alpha_{0}} + \frac{r_{\text{sr}}}{\beta_{0}} \right) W_{S}(\omega) \left| RA_{i}^{P} G_{\text{es1}}^{S} G_{\text{es2}}^{SS} G_{\text{es3}}^{S} E_{1} \right|^{2}
\]
(56)

\[
\langle P_{i}^{SPS}(x_{r},t|\omega) \rangle = \frac{1}{\rho_{0}} \sum_{S} \left( T_{\text{source}}, t - \frac{r_{\text{es1}}}{\beta_{0}} + \frac{r_{\text{es2}}}{\alpha_{0}} + \frac{r_{\text{sr}}}{\beta_{0}} \right) W_{S}(\omega) \left| RA_{i}^{P} G_{\text{es1}}^{S} G_{\text{es2}}^{SS} G_{\text{es3}}^{S} E_{1} \right|^{2}
\]
(57)

\[
\langle P_{i}^{SSP}(x_{r},t|\omega) \rangle = \frac{1}{\rho_{0}} \sum_{S} \left( T_{\text{source}}, t - \frac{r_{\text{es1}}}{\beta_{0}} + \frac{r_{\text{es2}}}{\alpha_{0}} + \frac{r_{\text{sr}}}{\beta_{0}} \right) W_{S}(\omega) \left| RA_{i}^{P} G_{\text{es1}}^{S} G_{\text{es2}}^{SS} G_{\text{es3}}^{S} E_{1} \right|^{2}
\]
(58)

\[
\langle P_{i}^{SSS}(x_{r},t|\omega) \rangle = \frac{1}{\rho_{0}} \sum_{S} \left( T_{\text{source}}, t - \frac{r_{\text{es1}}}{\beta_{0}} + \frac{r_{\text{es2}}}{\alpha_{0}} + \frac{r_{\text{sr}}}{\beta_{0}} \right) W_{S}(\omega) \left| RA_{i}^{P} G_{\text{es1}}^{S} G_{\text{es2}}^{SS} G_{\text{es3}}^{S} E_{1} \right|^{2}
\]
(59)

with

\[
A_{1} = \begin{pmatrix}
\hat{e}_{\text{es1}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{es1}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{es1}} \cdot \hat{e}_{\text{es3}} \\
\hat{e}_{\text{es2}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{es2}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{es2}} \cdot \hat{e}_{\text{es3}} \\
\hat{e}_{\text{es3}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{es3}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{es3}} \cdot \hat{e}_{\text{es3}}
\end{pmatrix}
\]
(60)

\[
A_{11} = \begin{pmatrix}
\hat{e}_{\text{sr1}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{sr1}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{sr1}} \cdot \hat{e}_{\text{es3}} \\
\hat{e}_{\text{sr2}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{sr2}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{sr2}} \cdot \hat{e}_{\text{es3}} \\
\hat{e}_{\text{sr3}} \cdot \hat{e}_{\text{es1}} & \hat{e}_{\text{sr3}} \cdot \hat{e}_{\text{es2}} & \hat{e}_{\text{sr3}} \cdot \hat{e}_{\text{es3}}
\end{pmatrix}
\]
(61)

\[
R = \begin{pmatrix}
\frac{\hat{e}_{\text{sr1}}(x)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{P(H)} & \frac{\hat{e}_{\text{sr1}}(x)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{SV(H)} & \frac{\hat{e}_{\text{sr1}}(x)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{SH(H)} \\
\frac{\hat{e}_{\text{sr1}}(y)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{P(H)} & \frac{\hat{e}_{\text{sr1}}(y)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{SV(H)} & \frac{\hat{e}_{\text{sr1}}(y)}{\sqrt{\hat{e}_{\text{sr1}}(x)^2 + \hat{e}_{\text{sr1}}(y)^2}} U_{SH(H)} \\
U_{P(V)} & U_{SV(V)} & 0
\end{pmatrix}
\]
(62)

A chain of characters such as PSS denotes the mode of the seismic waves propagating from the source to the reflection point to the scatterer and finally to the receiver. For example, PSS means that the source radiated P waves are converted to the S waves at the free surface and then S to S scattered by the scatterer. The matrix operator \(A_{1}\) and \(A_{11}\) are used for the transformation of coordinates from \(\hat{e}_{\text{es1}}, \hat{e}_{\text{es2}}, \hat{e}_{\text{es3}}\) to \(\hat{e}_{\text{sr1}}, \hat{e}_{\text{sr2}}, \hat{e}_{\text{sr3}}\) and from \(\hat{e}_{\text{es1}}, \hat{e}_{\text{es2}}, \hat{e}_{\text{es3}}\) to \(\hat{e}_{\text{sr1}}, \hat{e}_{\text{sr2}}, \hat{e}_{\text{sr3}}\). The matrix \(R\) is introduced for calculating the amplification of the free surface, where the term \(U_{SV(H)}\), for example, is the horizontal amplitude at the free surface for the incidence of the SV wave with unit amplitude (Appendix D). The letter in the parentheses following an orthogonal unit vector is used to denote the component of the Cartesian coordinates: e.g., \(\hat{e}_{\text{sr1}}(x)\) is the x component of \(\hat{e}_{\text{sr1}}\). We note that each of the Eqs. (48)–(59) ignores the phase information of the seismic waves propagating from the source to the receiver. Thus, if the phase of propagating seismic waves is altered by the free surface, we sum the P–SV and SH components not vectorially but separately to compute the power spectral density irrespective of the matrix operator \(R\).
From the discussion above, the power spectral density for the full set of scattered waves is written as

\[ P_{i}^{\text{SCATT}}(x_r, t|\omega) = P_{i}^{PP}(x_r, t|\omega) + P_{i}^{PS}(x_r, t|\omega) + P_{i}^{SP}(x_r, t|\omega) + P_{i}^{SS}(x_r, t|\omega) + P_{i}^{PPP}(x_r, t|\omega) + P_{i}^{PSP}(x_r, t|\omega) + P_{i}^{PSS}(x_r, t|\omega) + P_{i}^{PPP}(x_r, t|\omega) \]

This corresponds to the Eq. (40) for the inhomogeneous infinite media. The effect of the free surface makes the expression somewhat complicated. The power spectral densities for the direct P and S waves remain simple forms,

\[ P_{i}^{P}(x_r, t|\omega) = \frac{1}{\rho_0 \alpha_0} S \left( T_0, t - \frac{r_{ex}}{\alpha_0} \right) W_{p}(\omega) \left\| R G_{e}^{P} E_{i}^{P} \right\|^2 \]  

\[ P_{i}^{S}(x_r, t|\omega) = \frac{1}{\rho_0 \beta_0} S \left( T_0, t - \frac{r_{ex}}{\beta_0} \right) W_{s}(\omega) \left\| R G_{e}^{S} E_{i}^{S} \right\|^2 \]

where the matrix \( R \) is obtained by exchanging \( (\hat{e}_{x1}, \hat{e}_{x2}, \hat{e}_{x3}) \) that appeared in Eq. (62) with \( (\hat{e}_{x1}, \hat{e}_{x2}, \hat{e}_{x3}) \).

By using Parseval’s theorem in the same manner as in Eq. (41), we can construct the three-component seismogram envelopes in inhomogeneous semi-infinite media.

3. Simulation: the free-surface effect

By applying the method derived in the previous section, we can synthesize three-component seismogram envelopes for various sets of model parameters and source–receiver configurations. We use an inhomogeneous semi-infinite medium characterized by an exponential auto-correlation function with a correlation length \( a = 100 \) m and a fractional fluctuation strength \( \varepsilon = 4\% \). In investigating the effect of inhomogeneous characteristics on seismogram envelopes, we change the values of \( a \) and \( \varepsilon \). We divide the inhomogeneous media into many small cells with 200 m side length, and incoherently sum up the scattered waves from these

---

**Fig. 3.** Configuration of the earthquake source (fault plane) and the receivers located on the free surface. Locations of the receivers are shown in parentheses.
cells to construct seismogram envelopes. The size of the cell does not alter our results substantially but the temporal smoothness of envelope.

We place three receivers on the free surface of an inhomogeneous semi-infinite medium (Fig. 3). A detailed description for the locations of receivers is given in the figure. We assume a small local earthquake of $M_L = 2$ and calculate the three-component seismogram envelopes for the frequency band of 2–64 Hz. Predominant frequency of seismic waves is 13 Hz. The 1st, 2nd and 3rd components are parallel to the $x$, $y$ and $z$ axis, respectively. Synthetic envelopes are calculated from the source origin time to a 5-s lapse time. Because of the far-field approximation used in the synthetic formulas, we do not discuss the envelopes in the vicinity of the direct wave arrivals or amplitudes created by scatterers close to the source or the receivers. Thus, we do not include direct waves in the synthesis.

Fig. 4 shows three-component seismogram envelopes obtained at receiver R1 for a semi-infinite (solid curves) and infinite inhomogeneous medium (dashed curves). The amplitudes of the envelopes for the inhomogeneous infinite medium calculated from Eq. (41) are multiplied by a factor of two to incorporate the artificial free surface amplification, as commonly done in the past envelope studies (e.g., Sato, 1984). We see a

![Graph](image)

Fig. 4. Comparison of the synthetic velocity seismogram envelopes at receiver R1 in an inhomogeneous semi-infinite medium (solid curves) with those in an inhomogeneous infinite medium (dashed curves). We do not discuss envelope amplitude in the shaded interval. Synthesis is carried out in the frequency range of 2–64 Hz for a local earthquake with magnitude 2 and with the source parameters shown in Fig. 3. Parameters used here are as follows: $\alpha_0 = 6000$ m/s, $\beta_0 = 3464$ m/s, $\rho = 2700$ kg/m$^3$, $a = 100$ m, $e = 4\%$, $\nu = 0.8$, $L = 200$ m, and $\Delta T = 0.05$ s.
significant discrepancy between these model envelopes, especially in the portion of early S coda. In the 3rd component, the difference reaches about one half of their amplitude. This demonstrates the importance of calculating the free-surface effect on seismogram envelopes observed at a receiver located on the free surface.

Figs. 5 and 6 depict the effect of non-spherical source radiation on three-component seismogram envelopes. The former figure shows the envelopes obtained at receivers R1, R2 and R3 for the same earthquake, and the latter shows those obtained at R1 for three earthquakes of different focal mechanisms. The shape of envelope varies strongly depending on the focal mechanism and source–receiver configuration. The feature is most apparent in the early S coda, where the relation between the amplitude and the source radiation pattern is intricate. In the portion of the P coda, even at the receiver which is located in the nodal direction of P wave radiation (e.g., R1 in Fig. 5), envelope values of finite amplitude are found. Each component at the single station exhibits different and quite complicated envelope shapes (Fig. 6). Such differences are still found at large lapse times in our synthetic envelopes.

By fixing the source parameters, we construct the seismogram envelopes for different values of \( a \) and \( \varepsilon \). Fig. 7a shows the dependence of the seismogram envelopes on \( \varepsilon \). We find that the amplitude of the envelope is

---

Fig. 5. Three-component seismogram envelopes in an inhomogeneous semi-infinite medium. The envelopes obtained at receivers R1, R2 and R3 are shown. Parameters used are the same as those of Fig. 4.
enlarged with the increase of $\epsilon$. For the variation of the parameter $a$, we find the opposite tendency for the amplitude of the envelope (Fig. 7b). The amplitude is decreased with increasing $a$.

4. Discussion

The isochronal scattering shell represents the spatial distribution of the scatterers which contribute to the seismogram envelope at a fixed lapse time. For example, the isochronal scattering shells in an inhomogeneous infinite medium at a lapse time $t$ are obtained by substituting $S(T,0)$ into the Eqs. (28), (34)–(36). Fig. 8a schematically shows two dimensional cross sections of these shells. Travel times of the scattered waves associated with each point on a shell are by definition the same. Fig. 8b shows, on the other hand, isochronal scattering shells for inhomogeneous semi-infinite medium, where only source radiated S waves are considered. These shells are obtained from Eqs. (50), (51), (56)–(59). The conversion reflections at the free surface generate two additional isochronal scattering shells. In the same way, for source radiating P waves, the free surface modifies the shape of the isochronal scattering shells. The free surface reflection generates eight new isochronal scattering shells (e.g., PPP and SSS shells) in inhomogeneous semi-infinite media.
Fig. 7. Dependence of the seismogram envelope on the correlation length \( a \) and the rms fractional fluctuation strength \( \varepsilon \). Envelopes for the 1st component of receiver R1 are shown. Source parameters are the same as those for Fig. 4. (a) \( \varepsilon \) dependence. (b) \( a \) dependence.

In Fig. 9, we show the envelopes obtained at receiver R1 for 12 scattering modes in an inhomogeneous semi-infinite medium. We find that the envelopes for PP, PS and SP scattering modes have finite amplitudes only after the arrival time of the direct P waves. The onset of the envelope for SS scattering mode is found just after the arrival time of the direct S waves. In general, the amplitude of the envelope for SS scattering is dominant in the portion of the S coda. This is explained by the dominance of the source radiated energy of S waves in comparison to that of P waves, and by the effectiveness of SS scattering in comparison to SP scattering. In contrast, in the portion of P coda, we find SP scattering to be the dominant mode. As depicted in

Fig. 8. Isochronal scattering shells in inhomogeneous infinite media (a), and in inhomogeneous semi-infinite media (b), where only isochronal scattering shells for the source radiated S waves are shown. Solid circle and triangle indicate source and receiver location, respectively. Thin-line arrows schematically show ray paths of scattered waves: (a) PP, PS, SP and SS mode; (b) SSS mode.
Fig. 9. Envelope synthesis in an inhomogeneous semi-infinite medium. Three-component seismogram envelopes at receiver R1 for 12 scattering modes are shown. Parameters are the same as those used for Fig. 4.

Fig. 9, the amplitudes of seismogram envelopes corresponding to the scattering modes additionally generated by the free surface reflection are not small. Especially, the amplitude of the envelope for the SSS scattering mode is comparable to that of the SS scattering mode. The modification of the seismogram envelope amplitude in the early S coda as found in Fig. 4 is explained by the contribution of the newly generated SSS scattering mode. Whereas, envelope amplitude of P coda is not altered by the incorporation of the free surface.

As seen in Figs. 5 and 6, the effect of source radiation pattern on three-component seismogram envelopes is most apparent in the portion of early S coda. The dependence is rather intricate. On the other hand, the shapes of the envelopes corresponding to the P coda are quite similar at each receiver. For example, in Fig. 6, the amplitudes of the envelopes for the 2nd component are small for all focal mechanisms. This is explained, in connection with the dominance of the SP scattering in the P coda, by the fact that the SP scattering shell is not altered by the free surface for this lapse time and the ray paths of the SP scattered waves are nearly perpendicular to the y axis (2nd component). These characteristics found in SP scattering also imply that large parts of the scattered waves which consist of P coda are coming from the source direction. This feature of P coda waves is coincident with observational results reported from semblance and/or FK analysis for local earthquake seismograms (e.g., Wagner and Owens, 1993).

The variation in $a$-and $e$-values strongly affects the scattering strength and their direction pattern (see
Appendix A). For example, according to Sato (1984), the backward scattering strength for SS scattering depends on $\varepsilon^2/a$ as

$$g_{SS}(\omega) = \frac{(1 + \nu)^2}{2} \frac{\varepsilon^2}{a} \left(1 + \frac{A_0^2}{(4a^2\omega^2)}\right)^2$$

We find a spectral corner of $g_{SS}$ at $a\omega/\beta_0 \approx 2^{-1}$. For SS scattering caused by the inhomogeneities whose correlation length is much larger than the wavelength of the incident S waves ($a\omega/\beta_0 \gg 1$), the backward scattering intensity is proportional to $\varepsilon^2/a$. Thus, the amplitude of the envelope corresponding to S coda waves is roughly characterized by the ratio $\varepsilon^2/a$ as is shown in Fig. 7.

5. Conclusion

We presented a method of computing three-component seismogram envelopes of local earthquakes for a randomly inhomogeneous semi-infinite medium in the single scattering approximation. The specific features of this method are shown below.

1) Free surface reflections, including wave-type conversion and the incident angle dependence of the free surface amplification of seismic waves, are incorporated into the synthesis.

2) The effect of non-spherical source radiation on seismogram envelopes can be evaluated by using a double-couple point source.

3) The scattering amplitude is calculated on the basis of the Born approximation and the scattered waves of 12 scattering modes are taken into account.

The results obtained from the synthesis of three-component seismogram envelope in randomly inhomogeneous semi-infinite media are summarized below.

1) The free surface alters the shape of isochronal scattering shells and the amplitudes of incident scattered waves at a free-surface receiver. These effects are found in the amplitude of seismogram envelopes especially at the early S coda.

2) The source radiation pattern strongly affects the shapes of three-component seismogram envelopes; the relation between the amplitude of envelopes and the source radiation pattern is quite complicated; even at the receiver which is located in the nodal direction of P wave radiation, P coda envelopes have a finite amplitude.

3) S coda waves mostly consist of the SS scattered waves, whereas P coda waves are inclined to be dominated by the SP scattered waves; hence a large part of the scattered waves observed in the P coda are predicted to come from the source direction. P-coda envelope shape is not very sensitive to free-surface incorporation.

The synthetic method presented here provides a more accurate estimation of the stochastic inhomogeneous properties of the crust from observed seismogram envelopes of local earthquakes. Initial results from our approach are reported in a second paper (Yoshimoto et al., 1996) in this volume.

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Appendix A. Scattering of elastic waves

We use the Cartesian coordinate \( x = (x_1, x_2, x_3) \). We suppose that plane harmonic waves with the angular frequency \( \omega \) propagate in the positive \( x_m \) direction \( (m = 1, 2, 3) \) and impinge upon localized isotropic inhomogeneities which spread around the coordinate origin with a finite volume of \( L^3 \). The interaction between the incident plane waves and the inhomogeneities generates scattered spherical waves, which propagate outward from the origin of the coordinates. In the spherical coordinates, simple analytical expressions for the scattering amplitudes were obtained by Sato (1984) on the basis of the Born approximation with a correction for travel time fluctuation. Here, we present similar expressions described in the Cartesian coordinate system.

For the mathematical ease, we adopt the far-field approximation to the scattered waves. For the incidence of plane P waves with unit amplitude, the total wave field at a location \( x \) is written as

\[
u(x,t) = e^{i\omega t} \left( e^{i k_x x_1} \hat{x}_1 + \frac{e^{i k_x}}{r} F_{PP}(\hat{r}|\omega) \hat{x}_1 + \frac{e^{i k_y}}{r} F_{PS}(\hat{r}|\omega) \hat{x}_1 \right)
\]

where \( r = |x| \) and \( \hat{r} = x/r, k \) is the wavenumber of the P wave and \( l \) is that of the S wave. The symbol \( x_i \) is the directional unit vector in the \( i \)-th direction. The coefficients \( F \) are so called scattering amplitudes, where the superscripts denote scattering modes. On the other hand, the total wave field for the incidence of plane S waves with unit amplitude polarized in the \( x_n \) direction \( (n = 1, 2, 3 \text{ and } n \neq m) \) is written as

\[
u(x,t) = e^{i\omega t} \left( e^{i k_x x_n} \hat{x}_n + \frac{e^{i k_y}}{r} F_{SP}(\hat{r}|\omega) \hat{x}_n + \frac{e^{i k_x}}{r} F_{SS}(\hat{r}|\omega) \hat{x}_n \right)
\]

In these cases, scattering amplitudes for each scattering mode is expressed as

\[
F_{PP}(\hat{r}|\omega) = \frac{k^2 \hat{r}_i}{4\pi} \left[ \left( \hat{r}_m - 1 + \frac{2}{\gamma_0} (1 - \hat{r}_m^2) \right) \delta \tilde{\rho} (k\hat{r} - k\hat{x}_m) - 2 \frac{\delta \tilde{\alpha} (k\hat{r} - k\hat{x}_m)}{\alpha_0} \right] + \frac{4}{\gamma_0} \quad \quad (A.3)
\]

\[
F_{PS}(\hat{r}|\omega) = \frac{l^2}{4\pi} \left( \delta_{im} - r_i r_m \right) \left[ \left( \frac{2}{\gamma_0} \right) \frac{\delta \tilde{\rho} (l\hat{r} - k\hat{x}_m)}{\rho_0} - \frac{4}{\gamma_0} \frac{\delta \tilde{\beta} (l\hat{r} - k\hat{x}_m)}{\beta_0} \right] \quad \quad (A.4)
\]

\[
F_{SP}(\hat{r}|\omega) = \frac{k^2 \hat{r}_i \hat{r}_n}{4\pi} \left[ \left( \frac{2}{\gamma_0} \right) \frac{\delta \tilde{\rho} (k\hat{r} - l\hat{x}_m)}{\rho_0} - \frac{4}{\gamma_0} \frac{\delta \tilde{\beta} (k\hat{r} - l\hat{x}_m)}{\beta_0} \right] \quad \quad (A.5)
\]

\[
F_{SS}(\hat{r}|\omega) = \frac{l^2}{4\pi} \left[ \left( \delta_{im} - \hat{r}_i \hat{r}_n \right) \frac{\delta \tilde{\rho} (l\hat{r} - l\hat{x}_m)}{\rho_0} - 2 \left( \delta_{im} - \hat{r}_i \hat{r}_n \right) \frac{\delta \tilde{\beta} (l\hat{r} - l\hat{x}_m)}{\beta_0} \right] + cF_{SS} \quad \quad (A.6)
\]

with

\[
\gamma_0 = \frac{\alpha_0}{\beta_0} \quad \quad (A.7)
\]
where the tilde symbol represents the Fourier transform defined as follows: e.g.,

\[ \delta \alpha (x) = \frac{1}{(2\pi)^3} \int \int \delta \tilde{\alpha} (m) e^{im \cdot x} \, dm \]  

(A.8)

Additional terms for travel time correction in these equations are explicitly written as

\[ cF_i^{PP} (\hat{r} | \omega) = \frac{k^2 \hat{r}_i}{4\pi} \cdot (2\hat{r}_m) \cdot H(\psi_C - \psi) \frac{\delta \tilde{\alpha} (k\hat{r} - k\hat{k}_m)}{\alpha_0} \]  

(A.9)

\[ cF_i^{SS} (\hat{r} | \omega) = \frac{l^2}{4\pi} \cdot 2(\delta_{in} - \hat{\nu} \cdot \hat{\xi}_m) \cdot H(\psi_C - \psi) \frac{\delta \tilde{\beta} (l\hat{r} - l\hat{k}_m)}{\beta_0} \]  

(A.10)

where \( \psi = \cos^{-1} (\hat{r} \cdot \hat{x}_m) \) and \( \psi_C \) is the cut-off angle. \( H \) is the Heaviside step function. We note that the scattering amplitudes obtained here are slightly different from those obtained by Sato (1984). In the derivation of the scattering amplitudes, we added a condition that the second spatial derivatives of the travel time fluctuation can be negligible:

\[ \frac{\partial^2 (\delta \tau^P (x))}{\partial x_i \partial x_j} = \frac{\partial^2 (\delta \tau^S (x))}{\partial x_i \partial x_j} = 0 \]  

(A.11)

where \( \delta \tau^P (x) \) and \( \delta \tau^S (x) \) are the travel time fluctuation of P and S waves, respectively (see equations (14a) and (21a) in Sato, 1984). This condition means that the travel time fluctuation is a smooth function in space. Within the angle \( \psi_C \), the term \( H \) takes on non-zero value and suppresses the scattering amplitudes for the velocity fluctuation effectively. The direction dependence of the scattering amplitudes is quite different for each scattering mode. Assuming that the spatial velocity fluctuation of more than twice the wavelength of the incident waves causes travel time fluctuation, Sato (1984) used \( \psi_C = 2\sin^{-1}(1/4) \). The cut-off angle \( \psi_C \) has been studied in relation to scattering attenuation by many researchers (e.g. Chernov, 1960; Frankel and Clayton, 1986; Roth and Korn, 1993). Since the discussion about the cut-off angle as a whole is beyond the scope of this paper, we simply adopt \( \psi_C = 2\sin^{-1}(1/4) \).

Finally, in the stochastic sense, we obtain the mean-square scattering amplitudes caused by an inhomogeneous unit volume as follows:

\[ \langle |F_i^{PP} (\hat{r} | \omega)|^2 \rangle / L^3 = \frac{l^4}{(4\pi)^2} \Xi_{\xi} (k\hat{r} - k\hat{k}_m) \xi^{PP} (\hat{r})^2 \]  

(A.12)

\[ \langle |F_i^{PS} (\hat{r} | \omega)|^2 \rangle / L^3 = \frac{l^4}{(4\pi)^2} \Xi_{\xi} (l\hat{r} - l\hat{k}_m) \xi^{PS} (\hat{r})^2 \]  

(A.13)

\[ \langle |F_i^{SP} (\hat{r} | \omega)|^2 \rangle / L^3 = \frac{l^4}{(4\pi)^2} \Xi_{\xi} (k\hat{r} - k\hat{k}_m) \xi^{SP} (\hat{r})^2 \]  

(A.14)

\[ \langle |F_i^{SS} (\hat{r} | \omega)|^2 \rangle / L^3 = \frac{l^4}{(4\pi)^2} \Xi_{\xi} (l\hat{r} - l\hat{k}_m) \xi^{SS} (\hat{r})^2 \]  

(A.15)

where

\[ E_i^{PP} (\hat{r}) = \frac{\gamma_0^2}{\gamma_0^2} \left( \hat{r}_m - 1 + \frac{2}{\gamma_0^2} (1 - \hat{r}_m^2) \right) \nu - 2 + \frac{4}{\gamma_0^2} (1 - \hat{r}_m^2) + 2\hat{r}_m H(\psi_C - \psi) \right] \xi (k\hat{r} - k\hat{k}_m) \]  

(A.16)

\[ E_i^{PS} (\hat{r}) = (\delta_{in} - \hat{\nu} \cdot \hat{\xi}_m) \left[ \nu - 2 \frac{\gamma_0^2}{\gamma_0^2} \hat{r}_m \right] \xi (l\hat{r} - l\hat{k}_m) \]  

(A.17)
The directional dependence of the mean-square scattering amplitudes for a low angular frequency limit is mainly characterized by the term $E^2$. For a simple example of $R_{ij}$, let us suppose the exponential auto-correlation function represented by Eq. (7) in the text. In this case, we obtain the power spectral density functions that appeared in Eqs. (A.12), (A.13), (A.14) and (A.15) as

$$
\Xi_{\xi}(kF - k\xi_m) = \frac{8\pi e^2 \alpha^3}{1 + (4\alpha^2 \omega^2)/(\gamma_0^2 \beta_0^2) \sin^2(\psi/2)}
$$

$$
\Xi_{\xi}(lF - k\xi_m) = \Xi_{\xi}(kF - k\xi_m) = \frac{8\pi e^2 \alpha^3}{\left[1 + (\alpha^2 \omega^2)/(\gamma_0^2 \beta_0^2) \right]^2 (1 + \gamma_0^2 - 2\gamma_0 \cos \psi)}
$$

$$
\Xi_{\xi}(lF - \xi) = \Xi_{\xi}(lF - \xi) = \frac{8\pi e^2 \alpha^3}{\left[1 + (4\alpha^2 \omega^2)/(\beta_0^2) \sin^2(\psi/2)\right]^2}
$$

**Appendix B. Scattering attenuation**

Elastic waves in inhomogeneous media attenuate with increasing travel distance on account of scattering. This is so called scattering attenuation. The parameter $Q^{-1}$ representing the strength of scattering attenuation is directly estimated on the basis of the Born approximation in case that the fractional fluctuation of inhomogeneous media is weak. Sato (1984) derived following simple integral form for P and S wave scattering attenuation:

$$
Q_p^{-1}(\omega) = \frac{\alpha_0}{\omega} \Phi \left[ \frac{\left| \langle F_i^{PP}(\phi(\omega) \rangle \right|^2}{\gamma_0} \right] / L^3 \] d\Omega
$$

$$
Q_s^{-1}(\omega) = \frac{\beta_0}{\omega} \Phi \left[ \frac{\left| \langle F_i^{SS}(\phi(\omega) \rangle \right|^2}{\gamma_0} + \gamma_0 \left| \langle F_i^{SP}(\phi(\omega) \rangle \right|^2 \right] / L^3 \] d\Omega
$$

The integral in the right hand side of these equations is carried out over the whole solid angle. Both $Q_p^{-1}$ and $Q_s^{-1}$ are functions of the angular frequency.

**Appendix C. Explicit representation of matrices**

We list explicit description of the matrices for the scattering amplitude which is used for the synthesis of three-component seismic wave envelopes. Each matrix element appeared here is calculated from Eqs. (A.12), (A.13), (A.14) and (A.15) by letting $m = 3$. We denote the value of $n$ as the superscript in the parentheses.

$$
S^{PP} = \begin{pmatrix}
\sqrt{\left| \langle F_i^{PP}(\phi) \rangle \right|^2 / L^3} & 0 & 0 \\
\sqrt{\left| \langle F_i^{PP}(\phi) \rangle \right|^2 / L^3} & 0 & 0 \\
\sqrt{\left| \langle F_i^{PP}(\phi) \rangle \right|^2 / L^3} & 0 & 0
\end{pmatrix}
$$
Appendix D. Free surface effect

Coefficients for the reflection of the free surface are written as (see e.g., Aki and Richards, 1980; Kennett, 1991)

\[
R_{PP} = R_{SS(P - SV)} = \left[ -\left( \frac{1}{\beta_0^2} - 2p^2 \right) + 4p^2q_{\alpha_0}q_{\beta_0} \right] \Gamma^{-1}
\]  
(A.29)

\[
R_{PS} = \left[ 4\alpha_0\beta_0^{-1}pq_{\alpha_0} \left( \frac{1}{\beta_0^2} - 2p^2 \right) \right] \Gamma^{-1}
\]  
(A.30)

\[
R_{SP} = \left[ -4\alpha_0^{-1}\beta_0pq_{\beta_0} \left( \frac{1}{\beta_0^2} - 2p^2 \right) \right] \Gamma^{-1}
\]  
(A.31)

\[
R_{SS(SH)} = 1
\]  
(A.32)

where

\[
p = \begin{cases} 
\frac{\sin \theta_{es1}}{\alpha_0} & \text{for P wave incidence} \\
\frac{\sin \theta_{es1}}{\beta_0} & \text{for S wave incidence} 
\end{cases}
\]  
(A.33)

\[
q_{\alpha_0} = \sqrt{\frac{1}{\alpha_0^2} - p^2}
\]  
(A.34)
\[ q_{\beta_0} = \sqrt{\frac{1}{\beta_0^2} - \rho^2} \]

\[ I = \left( \frac{1}{\beta_0^2} - 2 \rho^2 \right)^2 + 4 \rho^2 q_{\alpha_0} q_{\beta_0} \]

In addition, coefficients for the amplification of the free surface are given as follows:

\[ U_{P(H)} = 4\alpha_0 \beta_0^{-2} pq_{\alpha_0} q_{\beta_0} I^{-1} \]

\[ U_{P(V)} = -2\alpha_0 \beta_0^{-2} q_{\alpha_0} \left( \frac{1}{\beta_0^2} - 2 \rho^2 \right) I^{-1} \]

\[ U_{SV(H)} = -2 \beta_0^{-1} q_{\beta_0} \left( \frac{1}{\beta_0^2} - 2 \rho^2 \right) I^{-1} \]

\[ U_{SV(V)} = -4 \beta_0^{-1} pq_{\alpha_0} q_{\beta_0} I^{-1} \]

\[ U_{SH(H)} = 2 \]

References


