Synthesis and applicable condition of vector wave envelopes in layered random elastic media with anisotropic autocorrelation function based on the Markov approximation

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SUMMARY
The Markov approximation is a powerful stochastic method for the direct synthesis of wave envelopes in random velocity fluctuated media when the wavelength is shorter than their correlation distance. To apply the Markov approximation to realistic cases, we consider horizontal layered random media characterized by an anisotropic autocorrelation function (ACF), where different layers have different randomnesses and different background velocities having step-like changes. Solving the parabolic master equation for the two frequency mutual coherence function (TFMCF) in random elastic media for the vertical incidence of a plane wavelet from the bottom, we calculate the angular spectrum just before the first velocity boundary. Multiplying transmission or PS conversion coefficients of the boundary by the angular spectrum, we calculate the angular spectrum and TFMCF on the other side of the boundary. Then we solve the master equation for the forward propagating wavelet in the second layer. Taking the same procedure for each layer boundary, we finally obtain the mean square (MS) vector envelopes on the free surface on the top layer. For the practical simulation, we use 2-D random media characterized by a Gaussian ACF. We numerically confirm the validity of the envelope synthesis for a specific case of layered random media with anisotropic ACF by comparing with finite difference (FD) simulations of elastic waves. Considering the Earth structure, the horizontal correlation distance is larger than the vertical one and the velocity fractional fluctuation becomes weak as depth increases, the Markov approximation is good for modelling the primary wavelet and also the converted wavelet for the vertical incident wavelet. We derive an applicable range of the Markov approximation for random media with anisotropic ACF by comparing with FD simulations. The results show that the Markov approximation is accurate when the wavelength is comparable or shorter than the both of vertical and horizontal correlation distances and the MS fractional fluctuation is much smaller than the ratio of squared horizontal correlation distance to the product of the vertical correlation distance and propagation distance.

Key words: Body waves, Computational seismology, Wave scattering and diffraction.

1 INTRODUCTION
Short period seismograms are complex reflecting highly heterogeneous structures in the Earth. Small scale heterogeneous structures have often been mathematically modelled by random inhomogeneities of elastic parameters. It is appropriate to use statistical methods to analyse short-period seismograms rather than deterministic ones. One of the statistical methods, the Markov approximation is a powerful method for the direct synthesis of wave envelopes in random media when the wavelength is shorter than the correlation distance of random inhomogeneities. The Markov approximation is a stochastic extension of the phase screen method, which was developed in studies about radio waves through the atmosphere where the dielectric constant is fluctuated and acoustic waves through the ocean disturbed by internal waves (e.g. Ishimaru 1978; Rytov et al. 1989). Extending the Markov approximation to vector waves, we can explain the broadening of seismogram envelopes of local earthquakes and the excitation of the transverse amplitude of P-waves of teleseismic events as scattering effects (e.g. Sato 2006, 2007; Kubanza et al. 2007). For the infinite uniform random medium case, Korn & Sato (2005) synthesized vector wave envelopes in 2-D infinite random media and confirmed the validity of the Markov approximation by comparing with the finite difference (FD)
simulations. These studies assumed uniform random media. In the real Earth, however, there are step-like velocity changes including the Moho as typically represented by the PREM (Dziewonski & Anderson, 1981). If we use the solution of the infinite uniform random medium model for the interpretation of observed seismogram envelopes, we misinterpret reflections and conversions at step-like velocity changes as scattering effects of the random inhomogeneity. It may lead to the overestimation of the strength of medium inhomogeneity.

Envelope syntheses based on the Markov approximation for non-uniform random media have been developed. Recently, Takahashi et al. (2008) used the stochastic ray path method for the envelope synthesis in layered random media, where each layer has different randomness. They used the Markov approximation for the mutual coherence function for stationary state and interpreted the histogram of accumulated traveltimes at a given receiver as the mean square (MS) envelope. Saito et al. (2008) synthesized envelopes in layered random media based on the Markov approximation by using the two frequency mutual coherence function (TFMCF) and compared those envelopes with the stochastic ray path method and the FD simulation in 2-D. These studies introduced step-like discontinuities in statistical parameters characterizing randomness, but the uniform background velocity for different layers. Emoto et al. (2010) succeeded in synthesizing the vector wave envelopes on the free surface of a uniform random medium for the vertical incidence of a plane wavelet. It is important to consider non-uniform random media having velocity discontinuities for realistic theoretical modelling; however, envelope syntheses in such media have not yet been succeeded.

Not only velocity discontinuities, but also anisotropies of random media are important. Because, we often see that the horizontal correlation distance is larger than the vertical one due to the geological evolution process of the Earth. For example, Nielsen et al. (2003) consider the depth dependent layered heterogeneous media with anisotropic autocorrelation function (ACF) to explain the observed characteristic of durations of coda waves of a peaceful nuclear experiment recorded by a long line array in Russia. Furumura & Kennet (2005) assumed that the oceanic slab is an anisotropic random medium with a longer correlation distance along the dip direction. Their FD simulation indicates that such a slab works as an effective guide of high frequency waves. Those studies suggest that it is necessary to take into account the anisotropic randomness in the Markov approximation to deal with such realistic problems. Sato (2008) introduced random media with anisotropic ACF as a target of the Markov approximation, and derived analytical solutions of vector wave envelopes for the case of anisotropic Gaussian ACF type random media. His derivation is restricted to the case when the global ray direction is parallel to one of the principle axes of anisotropic random media. The applicable condition of the Markov approximation for random media with anisotropic ACF has not been clarified yet, though that for isotropic random media has been discussed by comparing with FD simulations of elastic waves (Przybilla & Korn, 2008; Emoto et al. 2010).

Here, we derive vector wave envelopes for the following three cases based on the Markov approximation in 2-D. First, we synthesize envelopes in infinite random media with anisotropic ACF and examine the applicable range of the Markov approximation. Secondly, we synthesize envelopes on the free surfaces of a layered isotropic random medium and a layered anisotropic random medium as a realistic model. We also compare vector envelopes synthesized by using the Markov approximation for those random media with averaged vector envelopes derived by using FD simulations.

2 MARKOV APPROXIMATION FOR ANISOTROPIC RANDOM ELASTIC MEDIA

Here, we briefly summarize the theoretical scheme of the Markov approximation for random elastic media with anisotropic ACF according to Sato (2008).

2.1 Anisotropic random elastic media

In this study, we consider 2-D random media where the x- and z-axes are chosen in horizontal and vertical directions, respectively. We choose the positive z-direction to be upward. The P- and S-wave velocities are written as $\alpha(x) = \alpha_0(1 + \xi(x))$ and $\beta(x) = \beta_0(1 + \xi(x))$, respectively. Symbols $\alpha_0$ and $\beta_0$ are the average P- and S-velocities, respectively, and $\xi$ is a random function of a position $x = (x, z)$. We assume that the fractional fluctuation of the P- and S-wave velocities are the same and that of the mass density is proportional to the velocity fractional fluctuation (Birch’s law), where the proportional coefficient is chosen to be 0.8 (Sato & Fehler, 1998). We consider an ensemble of random functions ($\xi$), which are characterized by an anisotropic Gaussian ACF as $\langle \xi(x)\xi(x + \mathbf{x}) \rangle = \frac{\sigma_x^2}{\sqrt{\pi}} \exp(-x^2/\sigma_x^2) - \frac{\sigma_z^2}{\sqrt{\pi}} \exp(-z^2/\sigma_z^2)$, where $\sigma_x$ and $\sigma_z$ are correlation distances for the x- and z-directions, respectively. $\sigma$ is a root mean square fractional fluctuation of the velocity and the angular bracket means the ensemble average over random media. We also assume the small fractional fluctuation $|\xi| \ll 1$ and $\langle \xi \rangle = 0$.

2.2 Scheme of the Markov approximation

In the Markov approximation, we assume that the predominant wavelength ($\lambda_\perp$) is shorter than the correlation distance of the random fluctuation, that is $\lambda_\perp \leq \lambda_\parallel$ and $\lambda_\perp \leq \lambda_\parallel$. In such a case, narrow-angle scattering around the forward direction is dominant and the conversion scattering can be negligible. Therefore, we can treat P- and S-waves separately. We consider random elastic media where the velocity is fluctuated in $z \geq 0$ and homogeneous in $z < 0$. A vertically travelling plane P-wave enters the inhomogeneous medium from the bottom homogeneous medium. Since the scalar potential $\phi$ obeys the wave equation with velocity $\alpha(x, z)$, we expand $\phi$ by plane waves with angular frequency $\omega$ as

$$
\phi(x, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega z - i\omega t} \frac{1}{ik_0} U(x, z, \omega) \quad \text{for} \quad z > 0,
$$

where $k_0 = \omega/\alpha_0$. We define the TFMCF of $U(x, z, \omega)$ on the transverse line ($x$) as

$$
\Gamma_\perp(x', x'', z, \omega', \omega'') \equiv \langle U(x', \omega') U^* (x'', z, \omega'') \rangle.
$$

The TFMCF is a correlation of $U$ between different transverse coordinates $x'$ and $x''$ and different angular frequencies $\omega'$ and $\omega''$. By assuming that back-scattering can be negligible and waves are quasi-monochromatic with central angular frequency $\omega_c = (\omega' + \omega'')/2$, the TFMCF satisfies the following master equation (Sato, 2008):

$$
\partial_t \Gamma_\perp + i \frac{\omega_c}{2k_0^2 \alpha_0} \partial_x^2 \Gamma_\perp + k_0^2 \sigma_x^2 a_x \sqrt{\pi} \left( \frac{\sigma_x^2}{a_x^2} \right) \Gamma_\perp + \frac{\omega_c^2 \sigma_z^2 a_z \sqrt{\pi}}{2k_0^2} \Gamma_\perp = 0,
$$

where $a_x = \sqrt{\alpha_0/\beta_0}$ and $a_z = \sqrt{\alpha_0/\rho}$.
which can be simplified to
\[
\hat{\partial}_t \Gamma_2 + i \frac{\omega_d}{2k_c} \hat{\partial}_{x'}^2 \Gamma_2 + k_c^2 \varepsilon^2 a_s \sqrt{\pi} \left( \frac{x_0^2}{a_s^2} \right) \hat{\Gamma}_2 = 0,
\]
where \( \omega_d = \omega' - \omega'' \), \( k_c = \omega_d / \omega_0 \) and \( x' = x' - x'' \). The function \( \hat{\Gamma}_1 \) is the TMCF without the wandering term (Lee & Jokipii 1975a), where \( \Gamma_2 = \hat{\Gamma}_1 w(\omega_d) \). Function \( w(\omega_d) \) is \( \exp[-\omega_d^2 / (2a_s^2)] \) called the wandering term, which represents the traveltime (phase) fluctuation along the straight ray path. The stochastic derivation of eq. (4) is called the Markov approximation (Lee & Jokipii 1975b). In the derivation, for an increment \( \Delta \mathbf{z} \), we assume that \( \xi(t + \Delta \mathbf{z}) \) and \( \Delta \mathbf{z} \) are independent Gaussian random variables with zero mean and covariance,

\[
\langle \xi(t + \Delta \mathbf{z}) \xi(t) \rangle = \int d\mathbf{s} \int d\mathbf{s}' w(\mathbf{s} - \mathbf{s}') \mathbf{a}(s) \mathbf{a}^*(s').
\]

The intensity of waves is derived by using the TMCF. The \( x \)-component intensity is written as
\[
I_x = \langle \hat{\partial}_t \hat{\varphi}(x', z, t) \hat{\partial}_x^2 \phi(x'', z, t) \rangle_{x' = x''},
\]
and the \( z \)-component intensity is
\[
I_z = \langle \hat{\partial}_z \hat{\varphi}(x', z, t) \hat{\partial}_z \phi(x'', z, t) \rangle_{x' = x''}.
\]
In these equations, we introduce the intensity spectral density for each component, which are defined as
\[
\hat{I}_x(z, t, \omega_c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_d \, e^{-i\omega_d(t-\tau)} \left[ -\frac{1}{k_c^2} \hat{\partial}_z^2 \Gamma_2(x_d, z, \omega_c, \omega_d) \right]_{x_d = 0},
\]
and
\[
\hat{I}_z(z, t, \omega_c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_d \, e^{-i\omega_d(t-\tau)} \left[ \frac{1}{k_c^2} \hat{\partial}_z \Gamma_2(x_d, z, \omega_c, \omega_d) \right]_{x_d = 0}.
\]
Here, we approximated \( l/(k_c^2) \sim 1/k_c^2 \). Each component intensity spectral density corresponds to each component MS envelope at central angular frequency \( \omega_c \).

We introduce the reference MS envelope at a propagation distance \( z \) and central angular frequency \( \omega_c \) as
\[
\hat{I}^R(z, t, \omega_c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_d \, e^{-i\omega_d(t-\tau)} \Gamma_2(x_d, z, \omega_c, \omega_d).
\]
In the infinite medium, the reference MS envelope corresponds to the MS envelope of the sum of the \( x \)- and \( z \)-components, \( \hat{I}^R = \hat{I}_x + \hat{I}_z \). Replacing \( \Gamma_2 \) in (7), (8) and (9) with \( \hat{\Gamma}_2 \), we obtain MS envelopes without the wandering effect and write those as \( \hat{I}_0 \), \( \hat{I}_x \) and \( \hat{I}_z \), respectively.

The Fourier transform of the TMCF with respect to the transverse coordinate is called the angular spectrum, which represents the ray angle distribution of scattered waves on the transverse line at \( z \):
\[
\hat{\Gamma}_2(k_z, z, \omega_d, \omega_c) = \int_{-\infty}^{\infty} dx_d e^{-ik_z x_d} \Gamma_2(x_d, z, \omega_d, \omega_c).
\]
When the incident plane wavelet is impulsive \( \hat{I}_0(z, t, \omega_c) = \delta(t - z/\omega_0) \), that is, \( \hat{\gamma}_2 = \delta(x, z) \), the analytical solution of the TMCF is derived as equations (A7) and (A8a) in Sato (2008) in 3-D random media. We can calculate the MS envelopes in 2-D random medium by taking the limit as \( \omega_d / \omega_0 \to 0 \), where \( \omega_0 \) is a correlation distance of the \( x \)-component in expressions given by Sato (2008). The reference MS envelope without the wandering effect for the incidence of an impulsive plane \( P \)-wave case is
\[
\hat{I}_{0}^R(z, t, \omega_c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_d e^{-i\omega_d(t-\pi/2)} \frac{1}{\sqrt{\cos(\omega_c)}},
\]
which is the same as \( t_M \) in Sato (2008). The time integral of the reference MS envelope is constant at every travel distance:
\[
\int_{-\infty}^{\infty} dt \hat{I}_{0}^R = 1. \]

The longitudinal (vertical) component MS envelope is calculated by using the reference and transverse MS envelopes as
\[
\hat{I}_x(z, t, \omega_c) = \hat{I}_0^R(z, t, \omega_c) - \hat{I}_0(z, t, \omega_c).
\]

The envelopes are characterized by parameter \( t_M \) that contains the ratio of the correlation distance \( a_s / a_L^2 \). Scaling the time by using \( t_M \), we may write (11) and (13) as
\[
\hat{I}_0(z, t, \omega_c) = \frac{4\omega_d t_M}{\pi} \int_{-\infty}^{\infty} d(\omega_d t_M) \hat{e}^{-i\omega_d t_M} \left[ \frac{1}{\sqrt{\cos(\omega_c)}}, \right.
\]
and
\[
\hat{I}_x(z, t, \omega_c) = \frac{4\omega_d}{\pi} \left( \frac{t - z / \omega_0}{\omega_0} \right)^R \hat{I}_0(z, t, \omega_c).
\]

The amplitude of the transverse component is independent of \( t_M \) and that of the reference envelope is proportional to \( 1/t_M \). When the horizontal correlation distance is shorter than the vertical one, \( t_M \) becomes large so that the broadening effect becomes strong for both components and the excitation ratio of the transverse component relative to the longitudinal one becomes large. These characteristics of envelopes are the same as those for the 3-D case described in Sato (2008). The decays of amplitudes with propagation distance \( z \) of the reference and transverse component MS envelopes are proportional to \( z^{-2} \) and \( z^{-1} \), respectively. That of the longitudinal component MS envelope is nearly proportional to \( z^{-2} \), since the amplitude of the transverse component is small. For the real Earth structure, if the horizontal correlation distance is larger than the vertical one, then the envelopes for the vertical incidence of the plane \( P \)-wavelet become sharp and coda excitation is small compared with those for the horizontal propagation.

For a plane \( S \)-wavelet, we let the potential be \( B \), the propagation direction is \( z \) and the polarization axis is \( x \). Then the displacement vector can be written as \( u = (\partial_\varphi, \hat{\partial}_B, \hat{\partial}_B) \). We suppose the potential

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as the superposition of plane waves to be the same as (1), where \( k_0 = \omega \beta_0 \). We also define the TFMCF for the \( S \)-wave as in the eq. (2). Then the master equation of the TFMCF for quasi-monochromatic \( S \)-wave at \( \omega \), is

\[
\partial_x \Gamma_2 + \frac{\alpha_x}{2 \beta_x \beta_0} \partial_x \sigma \Gamma_2 + k_z^2 e^2 \alpha_z \sqrt{\pi} \left( \frac{x}{\alpha_z^2} \right) \partial_x \Gamma_2 = 0,
\]

where \( k_z = (\omega^2 + \omega^2) / \beta_x \) and definitions of other parameters are the same as \( P \)-wave case. Replacing \( x \) with \( z \) and \( \alpha_0 \) with \( \beta_0 \) in expressions of MS envelopes for \( P \)-wave case, (11), (13) and (14), we obtain those for the \( S \)-wave case.

### 2.3 Numerical tests of the applicable condition

We conduct FD simulations of wave propagation in anisotropic random media to examine the applicable range of the Markov approximation in 2-D by using the same method as used for the isotropic case (Emoto et al. 2010). The model space is 100 km in the \( x \)-direction and 146 km in the \( z \)-direction, where the medium is homogeneous for the bottom 19 km and random for the top 127 km. Our FD method uses the velocity-stress staggered grid scheme with forth order accuracy in space and second order in time. The grid space is about 98 m in both \( x \)- and \( z \)-direction and the time step is 5 ms. These settings satisfy the von Neumann’s stability condition (Moczo et al. 2000) and the grid dispersion is negligible. We set a periodic boundary condition at the left and right sides of the medium and the vertical incidence of a plane wavelet. We set a plane source wavelet in the bottom homogeneous layer. As a source wavelet, we use a Külpper wavelet whose dominant frequency is 2 Hz

\[
u(t, z) = \begin{cases} \sin \left[ 4\pi \left( t - \frac{z}{\omega} \right) \right] - \frac{1}{2} \sin \left[ 8\pi \left( t - \frac{z}{\omega} \right) \right] & 0 \leq t - \frac{z}{\omega} \leq 0.5 \text{s} \\ 0 & \text{otherwise.} \end{cases}
\]

This wavelet is a piece of a sine function and shows a simple and smooth shape (see Fig. 1). We put linear receiver arrays at travel distances of 25, 50, 75 and 100 km. Each receiver array consists of 20 receivers with an interval of 5 km. For Gaussian ACF type random media, the power spectrum is poor in small wavelengths, so the effect of the long scale heterogeneity is relatively large. We average squared waveforms, which are called FD envelopes, at 20 receivers to obtain one MS envelope in each array. We further conduct simulations for 100 realizations of anisotropic random media generated by different random seeds. Ensemble of 100 realizations of random media is enough to obtain a smooth envelope. We call the ensemble averaged envelope as the averaged FD envelope. For comparison, we convolve the MS envelopes derived by the Markov approximation, (13) and (14), with the wandering effect and the squared Külpper wavelet. The time integral of the Külpper wavelet is about 0.3. Therefore the time integral of the sum of convolved MS envelopes of \( x \) and \( z \)-components is 0.3 at every travel distance.

Fig. 1 compares the averaged FD envelopes (dashed black line) with the Markov envelopes (solid gray line) for the anisotropic random media, where the vertical correlation distance is shorter than the horizontal one, \( \alpha_x = 10 \) km and \( \alpha_z = 5 \) km. The shadow areas represent one standard deviations of FD envelopes. The Markov envelopes coincide well with the averaged FD envelopes in both two components at all distances. In Fig. 2, which is a magnified view of Fig. 1, we find a small difference between them that the averaged FD envelopes becomes a little large compared with the

![Figure 1. Comparison of Markov envelopes (solid gray curves) with averaged FD envelopes (dashed black curves) in infinite anisotropic random media at different travel distances for the vertical incidence of a 2 Hz plane P-wave. The random media are characterized by an anisotropic Gaussian ACF with \( \alpha_x = 10 \) km, \( \alpha_z = 5 \) km and \( \epsilon = 0.05 \), and \( \alpha_0 = 7.8 \) km s\(^{-1}\) and \( \beta_0 = 4.5 \) km s\(^{-1}\). Thick and thin curves indicate the longitudinal and transverse components, respectively. The Markov envelopes are convolved with the wandering effect and the squared Külpper wavelet. Gray area shows the standard deviation of FD envelopes. Characteristic times \( \tau_0 \) are \( 8.8 \times 10^{-3} \), \( 3.6 \times 10^{-2} \), \( 8.0 \times 10^{-2} \) and 0.14 s at \( z = 25, 50, 75 \) and 100 km, respectively. In the right box, we show a 2 Hz Külpper wavelet (solid) and its squared trace (dash). We also show the schematic illustration of a random medium and the vertical incidence of a plane wavelet.](image507_to_535x649)

![Figure 2. Magnified view of Fig. 1.](image507_to_535x689)

Markov envelopes about 1 to 1.5 s after the onset. This is because, the Markov approximation neglects the back-scattering, while the FD simulation takes into account it, so the averaged FD envelopes have relatively large coda waves.

We compare the Markov envelopes with the averaged FD envelopes in anisotropic random media where the horizontal correlation distance is shorter than the vertical one, \( \alpha_x = 5 \) km and \( \alpha_z = 10 \) km in Fig. 3. The shapes of the Markov envelopes and that of the averaged FD envelopes roughly coincide with each other, though there are some discrepancies in their amplitudes. The discrepancies are larger than those of the case of Fig. 1 in both components at all travel distances. The longitudinal component of the Markov envelopes are smaller than those of the averaged FD envelope while the transverse component of the Markov envelopes are larger than those of the averaged FD envelopes. For the case of Fig. 3, \( \tau_0 \) becomes large compared with the case of Fig. 1, so the excitation of the transverse component and the broadening effect of the both two components are large. In Fig. 1, the time integral of the sum of the two components of the averaged FD envelope is constant at every travel distance, but that for the case of Fig. 3 beyond the initial value. This means that the large angle scattering occurs in the case of Fig. 3, so the Markov approximation does not work well. For the Markov approximation, the following conditions should be satisfied, \( \alpha_x k_x \gg 1 \), \( \alpha_z k_z \gg 1 \)
and $\varepsilon^2 a z / a_0^2 \ll 1$. First and second conditions are necessary for the forward-scattering regime. Third condition indicates that the wavelength is shorter than the coherence radius which is defined as $a_0 = a / \sqrt{\pi \varepsilon^2 a_z^2 z}$ (Sato, 2008). This condition means that the excitation of the transverse component should be small compared with the longitudinal component (Sato, 2008). At a travel distance of 100 km, $a_a k_0 = 16$, $a_a k_0 = 8.1$, $\varepsilon^2 a_z a_z^2 = 0.0125$ for the case of Fig. 1 and $a_a k_0 = 8.1$, $a_a k_0 = 16$, $\varepsilon^2 a_z a_z^2 = 0.1$ for the case of Fig. 3. It means that the third condition is more critical for the applicable condition for random media with an anisotropic ACF.

We calculate the Markov envelopes at a propagation distance of 100 km for various parameter sets of $a_a$ and $a_0$ (see Fig. 4). We change $a_a$ and $a_0$ as 2.5, 5.0 and 10 km for the vertical incidence of an impulsive plane $P$-wavelet. When $a_a$ becomes large, Markov envelopes become broad due to the large $t_M$. For $(a_a, a_0) = (2.5, 2.5)$ km, (2.5, 5.0) km and (2.5, 10) km, amplitudes and peak arrival times are different between the Markov and averaged FD envelopes. For $(a_a, a_0) = (5.0, 10)$ km, the peak arrival times are the same each other, but the amplitudes are different. For $(a_a, a_0) = (5.0, 5.0)$ km, the Markov envelope is slightly smaller than the averaged FD envelope at the peak: $I_{M \text{max}}^F - I_{M \text{max}}^F = 11$ per cent, where $I_{M \text{max}}^F$ and $I_{M \text{max}}^F$ are maximum values of the longitudinal component MS envelopes derived by the Markov approximation and that derived by the FD simulation, respectively. For $(a_a, a_0) = (10, 2.5)$ km, the Markov envelope is slightly larger than the averaged FD envelope at the peak: $|I_{M \text{max}}^F - I_{M \text{max}}^F| / I_{M \text{max}}^F = 11$ per cent. Considering these results, we conclude that the applicable condition of the Markov approximation in the case of anisotropic Gaussian ACF is as follows with an accuracy of 11 per cent

$$\begin{align*}
a_a k_0 &\geq 4.0 \\
0.05 \varepsilon^2 a_z^2 &\leq 8.1 \\
100 &\ll \varepsilon^2 a_z^2 \ll 1.
\end{align*}$$

(19)

For the isotropic half-space case, Emoto et al. (2010) suggested that $a_a k_0 \geq 8$ and $\varepsilon^2 \leq (0.1 \sim 0.2)a_z^2$ are needed for the Markov approximation, where $a$ is the correlation distance. When the horizontal correlation distance is shorter than the vertical one, the Markov approximation does not work well. Therefore, condition (19) is more strict condition compared with that of the isotropic case. In the case of $(a_a, a_0) = (5.0, 10)$ km, $a_a k_0 = 16$, $a_a k_0 = 8.1$ and $\varepsilon^2 a_z a_z^2 = 0.1$, the Markov and averaged FD envelopes show the large discrepancy.

3 MARKOV APPROXIMATION FOR LAYERED RANDOM ELASTIC MEDIA

We consider a 2-D layered random medium as shown in Fig. 5. The average velocities in different layers are different.

3.1 Scheme of the Markov approximation

We assume that the statistical parameters are the same in both two layers. Although this condition is not necessary for our method, we focus on the effect of the step-like velocity structure in this study. Effects of the discontinuities of statistical parameters have been already discussed in Saito et al. (2008). We take into account the conversion of $PP$, $PS$, $SP$ and $SS$ at the velocity steps between the layers, but no conversion occurs within each layer. Here, we explain the procedure taking a two-layered isotropic random medium with the correlation distance $a$ on a homogeneous medium for simplicity. The velocity step and the free surface exist at $z = z_1$ and $z_2$, respectively, and we synthesize the MS envelopes on the free surface of the random medium. The procedure consists of four steps: at the bottom layer, the layer boundary, the upper layer and the free surface.

(i) Bottom layer ($0 \leq z \leq z_1$): We analytically solve the master eq. (4) for the $P$-wave incidence or (17) for the $S$-wave incidence in the bottom layer and obtain the TFMCF $o \Gamma_z(x_d, z_1, w_0)$ just beneath the layer boundary $z = z_1$ (Korn & Sato, 2005). We assume that the initial condition is an impulsive plane wavelet: $o \Gamma_z(x_d, z = 0, w_0) = 1$.

(ii) Layer boundary ($z = z_1$): We calculate the angular spectrum, eq. (10), by using the FFT. By taking the inverse Fourier transform of the angular spectrum with respect to angular frequency, we obtain the angular spectrum in the time domain:

$$I_0^A(k_z, z_1, t, \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw_a e^{-i(w_0 \pi - \pi i/w_0)} o \Gamma_z(k_z, z_1, \omega_0, \omega_0),$$. (20)

where $\omega_0$ is the average $P$-wave velocity in the bottom layer. For the $S$-wave case, we replace the $P$-wave velocity with the $S$-wave velocity. We show the angular spectrum in the time domain for the incidence of an impulsive plane $P$-wavelet case in Fig. 6 with the reference MS envelope calculated by the integration of the time domain angular spectrum $I_0^A$ over all wavenumbers (Korn & Sato, 2005). The ray angle can be calculated as $\theta = \sin^{-1}(k_z/k_0)$, where $k_z = \omega_0 / \alpha_0$ and $\omega_0 / \beta_0$ for $P$- and $S$-waves, respectively. $\beta_0$ is the average $S$-wave velocity in the bottom layer. While the angular spectrum has a peak value at 0 degrees and shows a sharp distribution just after the onset, the peak angle shifts to a wide angle, 9.6° at 0.08 s, and the distribution becomes flat and small with increasing reduced time. Even at 0.05 s, however, the distribution is still concentrated within 30°. This means that the later part of the envelope consists of large-angle scattered waves, but most of energies are carried by narrow-angle scattered waves. Since $\lambda_0 < a$, we may regard each scattered wave as a plane wave in the vicinity of the boundary, we multiply the angular spectrum $o \Gamma_z^A(x_d, z_1, \omega_0, \omega_0)$ by the squared transmission coefficient: $|\hat{P} P(k_z)|^2$ for $P$ to $P$ transmission wave, $|\hat{P} S(k_z)|^2$ for $P$ to $S$ conversion wave, $|\hat{S} S(k_z)|^2$ for $S$ to $S$ transmission wave or $|\hat{S} P(k_z)|^2$ for $S$ to $P$ conversion wave. Symbol $z_1$ means the position just below the boundary of $z_1$. The multiplied angular spectrum is represented as $o \Gamma_z^A(k_z, z_1, \omega_0, \omega_0)$.
Figure 4. Comparison of Markov envelopes (solid gray) and averaged FD envelopes (dashed black) in infinite anisotropic random media at $z = 100$ km for different sets of $a_x$ and $a_z$ for the vertical incidence of a 2 Hz plane $P$-wavelet (Kipper type) which has only the vertical component, where $\alpha_0 = 7.8$ km s$^{-1}$, $\beta_0 = \alpha_0/\sqrt{3} = 4.5$ km s$^{-1}$ and $\varepsilon = 0.05$. Thick and thin curves show the longitudinal and transverse components, respectively.

Figure 5. Model setting of the layered random media. (a) An example of the two-layered random medium characterized by a Gaussian ACF, where $a = 10$ km and $\varepsilon = 0.05$. The thickness of each layer is 50 km and there is a homogeneous layer below the random medium. Periodic and absorbing boundary conditions are implemented at the both sides and the bottom of the medium, respectively. (b) An example of the vertical velocity profile, where solid and dashed curves indicate the $P$- and $S$-wave velocities, respectively.
Figure 6. (a) Snapshots of the angular spectrum \( \alpha^A_{\theta} \) in the infinite random media calculated by the Markov approximation at \( z_1 = 50 \text{ km} \) for different reduced times, 0.01 s, 0.03 s and 0.05 s, where ray angle \( \theta = \sin^{-1}(k_x/k_z) \). The random media are characterized by \( \alpha_{10} = 7.8 \text{ km}^{-1} \), \( \beta_{10} = 4.5 \text{ km}^{-1} \), \( \rho_{10} = 3000 \text{ kg/m}^3 \), \( \sigma_2 = 6.0 \text{ km}^{-1} \), \( \beta_{20} = 3.5 \text{ km}^{-1} \), \( \rho_{20} = 2446 \text{ kg/m}^3 \) and a Gaussian ACF with \( a = 10 \text{ km} \) and \( \epsilon = 0.05 \). Solid lines indicate the angular spectrum just below the layer boundary at 50 km and dashed lines indicate those just above the layer boundary after the correction using transfer coefficients. (b) The reference MS envelope \( I_{0,0}^A \) just below the layer boundary (solid) and just above the layer boundary (dashed) without the wandering effect, which are calculated by integrating the angular spectrum with respect to the wavenumber.

where \( z_{\text{low}} \) is the position just above the boundary:

\[
\begin{align*}
0 \tilde{\Gamma}_2(k_x, z_{\text{low}}, \omega_j, \omega_e) &= \left\{ \begin{array}{ll}
\tilde{P}(k_x) \tilde{P}(k_x) & \text{for } P \text{ to } P \\
\tilde{P}(k_x) \tilde{S}(k_x) & \text{for } P \text{ to } S \\
\tilde{S}(k_x) \tilde{S}(k_x) & \text{for } S \text{ to } S \\
\tilde{S}(k_x) \tilde{P}(k_x) & \text{for } S \text{ to } P
\end{array} \right.
\end{align*}
\]

The transmission coefficients at the velocity boundary are taken from Aki & Richards (2002).

(iii) Upper layer \( (z_1 \leq z \leq z_2) \): We take the inverse Fourier transform of the angular spectrum \( \tilde{\alpha}^A_{\theta} \tilde{\Gamma}_2(k_x, z_{\text{low}}, \omega_j, \omega_e) \) with respect to the wavenumber to obtain the TFMCF in the space domain \( \alpha^A_{\theta} \Gamma_2(x_d, z_{\text{low}}, \omega_j, \omega_e) \). By using \( \alpha^A_{\theta} \Gamma_2(x_d, z_{\text{low}}, \omega_j, \omega_e) \) as the initial condition, we numerically solve the master equation in the upper layer until the free surface, \( \alpha^A_{\theta} \Gamma_2(x_d, z = z_2, \omega_j, \omega_e) \). Since we neglect the \( P-S \) and \( S-P \) conversion scatterings in the propagation within each layer, we solve the master equations for \( P-S \) and \( S-S \)-waves, independently. Here, \( k_x = \omega_c/\alpha_{20} \) and \( \omega_c/\beta_{20} \) for \( P \)- and \( S \)-wave, respectively, where \( \alpha_{20} \) and \( \beta_{20} \) are average \( P \)- and \( S \)-wave velocities in the upper layer, respectively. By using these definition of \( k_x \), we intrinsically take into account the refraction of rays at the boundary due to Snell’s law.

There is no analytical solution of the master eq. (4) or (17) except for the incidence of an impulsive wavelet; therefore, we use the Crank-Nicholson method to solve numerically those equations in the above layer. To this end, we need the TFMCF which is sampled evenly in space. So we do not use the normalized variable as used in Saito et al. (2002) to solve the master equation. We use the following boundary condition:

\[
\begin{align*}
\frac{\partial \tilde{\Gamma}_2}{\partial x_d} |_{x_d = 0} &= 0, \\
\frac{\partial \tilde{\Gamma}_2}{\partial x_d} |_{x_d = X_d} &= 0,
\end{align*}
\]

where \( X_d \) is the maximum value of \( x_d \) in the numerical calculation.

We check the validity of this boundary condition by comparing MS envelopes derived by using the Crank-Nicholson method with those of the analytical solution of the MS envelopes in the infinite isotropic Gaussian ACF random media as derived by Korn & Sato (2005). We assume \( a = 5 \text{ km} \), \( \epsilon = 0.05 \), \( z = 50 \text{ km} \) and \( \alpha_0 = 7.8 \text{ km}^{-1} \) and the vertical incidence of an impulsive plane \( P \)-wavelet. We set \( \Delta x_d = 0.05 \text{ km} \), \( \Delta z = 0.1 \text{ km} \), \( \Delta w_d = 0.2 \text{ s}^{-1} \) and \( x_p = 50 \text{ km} \). The differences between the analytical solution and the numerical result are only 1.4 per cent and 0.7 per cent at the peak value of longitudinal and transverse MS envelope, respectively. Under the boundary condition (22), the coefficient matrix appeared in the Crank-Nicholson method is tridiagonal, so the numerical calculation is fast.

(iv) Free surface \( (z = z_2) \): Just beneath the free surface, we calculate the angular spectrum by using FFT, \( \tilde{\alpha}^A_{\theta} \tilde{\Gamma}_2(k_x, z_2, \omega_j, \omega_e) \). Multiplying it by the squared plane wave amplification factor of the vertical component, \( |\varphi(k_x)|^2 \), or that of the horizontal component, \( |\alpha_0(k_x)|^2 \), we obtain MS envelopes on the free surface as proposed by Emoto et al. (2010). When we calculate the \( S \)-wave envelope in the upper layer, we use the squared amplification factors for the incidence of plane \( S \)-wave. The horizontal component MS envelope on the free surface without the wandering effect is

\[
I_{0,0}^\text{Free}(z_2, t, \omega_e) = \frac{1}{2\pi} \int_{-k_c}^{k_c} dk_x |\varphi(k_x)|^2 I_0^A(k_x, z_2, t, \omega_e),
\]

and the vertical component MS envelope is

\[
I_{0,0}^\text{Free}(z_2, t, \omega_e) = \frac{1}{2\pi} \int_{-k_c}^{k_c} dk_x |\alpha_0(k_x)|^2 I_0^A(k_x, z_2, t, \omega_e),
\]

where \( k_c = \omega_c/\alpha_{20} \) and \( \omega_c/\beta_{20} \) for \( P \)-wave and \( S \)-wave in the upper layer, respectively. The integral intervals of (23) and (24) are from \( -k_c \) to \( k_c \) to keep the physical meaning of the angular spectrum. To compare with FD simulations in the following subsection, we need to convolve eqs (23) and (24) with the wandering term and the squared source wavelet. For multi-layer model, we multiply the wandering terms of all layers taking into account the \( P \)- and \( S \)-wave propagations in the frequency domain. For example, the wandering
effect for $P$ to $P$ transmitted wave of the two-layer model is written as

$$w^P_{\text{MS}}(\omega_0) = \exp \left[ -\frac{\omega_0 \beta^2 a \sqrt{\pi}}{2} \left( \frac{z_1 + z_2 - z_1}{\alpha^2 z_0} \right) \right], \quad (25)$$

and that for $P$ to $S$ converted wave is written as

$$w^S_{\text{MS}}(\omega_0) = \exp \left[ -\frac{\omega_0 \beta^2 a \sqrt{\pi}}{2} \left( \frac{z_2 - z_1}{\alpha^2 z_0} \right) \right]. \quad (26)$$

The wandering effect takes into account the broadening due to the traveltime fluctuation. By convolving $I_{\text{Free}}$ and $I_{\text{Free}}$ with the wandering term, we obtain the MS envelopes with the wandering effect. To make envelopes that take into account later phases, for example, $P$–$P$ transmitted and $P$–$S$ converted waves, we calculate the $P$–$P$ and $P$–$S$ envelopes separately and combine them after convolved with the wandering effect and the squared source wavelet.

In the case of multilayered random media more than two, we repeat (ii) and (iii) until the waves reach the free surface. We have explained only the upward transmitted and converted waves to synthesize the envelopes on the free surface, but we can also calculate the downward reflected and converted waves by multiplying the angular spectrum by the reflection and conversion coefficients for plane waves at the velocity steps.

We show the MS envelopes without the wandering effect of $P$ to $P$ transmitted waves calculated by using the above procedure in Fig. 7. In the range of the reduced time from 0.0 to 1.0 s, $P$–$S$ converted waves at the boundary do not reach the free surface. For comparison, envelopes on the free surface of three types of single-layer random medium whose average $P$-wave velocities are 7.8, 6.9 and 6.0 km s$^{-1}$, respectively, are shown. All random media are characterized by a Gaussian ACF with $a = 10$ km and $\varepsilon = 0.05$. Comparing with the envelopes of single-layer models, those of two-layer model are amplified due to the low velocity of the upper layer. The excitation of the horizontal component relative to the vertical component of the two-layer model is 10 per cent smaller than those of single-layer models at the peak values. We can interpret the fact as follows: scattered waves are once spread in the bottom layer, but their rays are bent to the vertical direction at the velocity boundary, so the angular spectrum becomes sharp around the vertical direction and the relative excitation of the horizontal component on the free surface is suppressed. Note that the wandering effects are different for different random media. From eqs (25) and (26), we find that the broadening effect of the wandering for $P$ to $P$ transmitted wave is smaller than that for $P$ to $S$ converted wave.

### 3.2 Comparison with FD simulations for layered isotropic random elastic media

We examine the validity of the Markov approximation for the layered isotropic random media by using FD simulations. The medium setting is shown in Fig. 5, which is used for the calculation of Fig. 7. We set the free surface boundary condition at the top of the layered random medium. The source wavelet is a 2 Hz plane Képpler wavelet. We put 20 receivers on the free surface with an interval of 5 km and average the records of all receivers. To obtain a smooth envelope, we conduct FD simulations for 100 random media which are characterized by the same average velocities and statistical parameters with different randomnesses. Fig. 8 shows averaged FD envelopes on the free surface of the layered random medium for the vertical incidence of a plane $P$-wavelet. We also show the Markov envelopes that are derived based on (23) and (24) and then convolved with the wandering effect and the Képpler wavelet. The Markov envelopes and averaged FD envelopes well coincide each other in both of the two components from the onset to the coda part of the envelope. In the horizontal component, large amplitude seen at about 14.7 s in lapse time is $P$ to $P$ transmitted wave at the layer boundary, and $P$ to $S$ converted wave at the boundary arrives at about 20.8 s, about 6.1 s after the onset of $P$ to $P$ transmitted waves. For $P$ to $P$ transmitted wave, $|I^P_{\text{x max}} - I^P_{\text{min}}|/I^P_{\text{max}} = 5.9$ per cent and $|I^P_{\text{x max}} - I^P_{\text{min}}|/I^P_{\text{max}} = 0.6$ per cent, where subscript $x$ denotes the horizontal component. For $P$–$S$ transmitted waves, $|I^S_{\text{x max}} - I^S_{\text{min}}|/I^S_{\text{max}} = 15$ per cent, where $I^S_{\text{max}}$ means the peak value of the horizontal component of $P$–$S$ transmitted wave envelope. The difference of peak values of the Markov and averaged FD envelopes in the horizontal components of $P$–$P$ and $P$–$S$ envelopes are less than 0.1 per cent, which is quite small compared with the peak value of the vertical component of $P$–$P$ envelope.
3.3 Comparison with FD simulations for layered anisotropic random media

We consider another example: 2-D two-layered random media, both layers are characterized by different anisotropic Gaussian ACFs (see Fig. 9). Some previous studies show that the shallow structure is more heterogeneous than deeper one in the real Earth [e.g. Flatt & Wu (1988), Lee et al. (2003)]. So we set \( \alpha_{1} = 7.8 \text{ km s}^{-1},\ \alpha_{2} = 10 \text{ km},\ \alpha_{3} = 5 \text{ km} \) and \( \epsilon = 0.03 \) in the bottom layer. In the upper layer, \( \alpha_{1} = 6.0 \text{ km s}^{-1},\ \alpha_{2} = 5.0 \text{ km},\ \alpha_{3} = 2.5 \text{ km} \) and \( \epsilon = 0.05 \). S-wave velocities are \( 1/\sqrt{3} \) times of the P-wave velocities. We assume that a 2Hz plane Kummer wavelet vertically enters the random media from the bottom homogeneous layer as a P-wave source. Fig. 10 compares the Markov envelopes with the averaged FD ones on the free surface of the layered anisotropic random media. We can see that both envelopes well coincide each other from the onset to around the peak. But in the coda part of the envelope of the P to P transmitted waves at the layer boundary, from about 15.5 s, the averaged FD envelope starts to deviate from the Markov envelope. In this case, \( I_{M} \) is smaller than the case shown in Fig. 8 due to the short vertical correlation distance and the Markov envelope is sharp. So the Markov approximation for the coda part of the envelope does not work well in this case. For P-P transmitted waves, \( |I_{M} - I_{F}|/I_{F} = 5.2 \) per cent and \( |I_{M} - I_{F}|/I_{F} = 21 \) per cent. However, the absolute value of horizontal component is small compared with the vertical component, that is \( |I_{M} - I_{F}|/I_{F} = 0.1 \) per cent. For P-S transmitted waves, \( |I_{M} - I_{F}|/I_{F} = 42 \) per cent. This difference is less than 0.1 per cent of the maximum value of

Figure 8. (a) Comparison of Markov envelopes (solid gray curves) with those of averaged FD envelopes (dashed black curves) on the free surface of the two-layered isotropic random medium as shown in Fig. 5 for the vertical incidence of a 2 Hz plane P wavelet (Kummer type). Markov envelopes are convolved with the wandering effect and the squared Kummer wavelet. Thick and thin curves indicate the vertical and horizontal components, respectively. Gray area shows the standard deviation of FD envelopes. (b) and (c) are magnified view of the horizontal component.

Figure 9. Model setting of the layered anisotropic random media. (a) An example of the two-layered anisotropic random media. Both random layers are characterized by an anisotropic Gaussian ACF. The thickness of each layer is 50 km and there is a homogeneous layer below the random medium. In the bottom random layer, \( \alpha_{1} = 10 \text{ km},\ \alpha_{2} = 5.0 \text{ km},\ \epsilon = 0.03 \). \( \alpha_{10} = 7.8 \text{ km s}^{-1},\ \beta_{10} = 4.5 \text{ km s}^{-1} \) and \( \rho_{10} = 3000 \text{ kg m}^{-3} \). In the upper random layer, \( \alpha_{1} = 5.0 \text{ km},\ \alpha_{2} = 2.5 \text{ km},\ \epsilon = 0.05 \). \( \alpha_{20} = 6.0 \text{ km s}^{-1},\ \beta_{20} = 3.5 \text{ km s}^{-1} \) and \( \rho_{20} = 2446 \text{ kg m}^{-3} \). (b) An example of the vertical velocity profile.

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the vertical component of $P$–$P$ envelope. There is an offset between the averaged FD and the Markov envelopes before the onset of $P$–$S$ converted envelope due to wide-angle scattered waves of $P$–$P$ transmitted waves. If we remove this offset and fit the amplitude of the Markov envelope with that of the averaged FD envelopes at the onset of $P$–$S$ converted envelope, the Markov approximation for $P$–$S$ converted wave works well in terms of the amplitude and the duration of the envelope. For the modelling of later phases, we have to pay attention to the coda amplitude of the primary wave.

4 SUMMARY

We have synthesized vector wave envelopes in the 2-D infinite random elastic media with anisotropic ACF based on the Markov approximation as one of the realistic model and examined the applicable condition of it by comparing with the FD simulations of elastic wave propagation. We have found that the horizontal correlation distance plays a key role for determining the applicable range of the Markov approximation for the vertical incidence of a plane wavelet. Our simulations show that the Markov approximation is valid when $a \kappa_z \geq 8.1$, $a \kappa_x \geq 4.0$ and $\varepsilon^2 a_z a_x^2 \leq 0.05$.

We have developed the Markov approximation and synthesized the MS envelopes on the free surface of the layered random media for the vertical incidence of a plane wavelet, where step-like velocity discontinuities exist. We have focused on the angular spectrum that describes the ray angle distribution of scattered waves on the transverse line. We have succeeded in modelling the amplification and refraction at the layer boundary due to the existence of the upper low velocity layer. By comparing the MS envelopes derived based on the Markov approximation with those derived by using FD simulations, both envelopes show good coincidence from the onset of the $P$ to $P$ transmitted wave at the boundary to the coda part of it. We can also calculate the envelope of $P$ to $S$ converted waves at the boundary with enough accuracy.

We have calculated only the envelopes in 2-D random elastic media, but we can extend to the 3-D case by using the same procedure as we proposed in this paper. It is necessary to perform many FD simulations to obtain a smooth average MS envelopes, but it costs a lot especially in 3-D case; therefore we have examined the validity of the stochastic method in 2-D. We did not change the propagation distance when we examined the applicable condition of the Markov approximation for the random media with anisotropic ACF. It will be necessary to check the dependence of the propagation distance for applying the Markov approximation to layered random media where various layer thickness exist. Since our method is restricted for the case of the vertical incident plane wave, it will be necessary to extend the method to the oblique incidence case. We are planning to distinguish medium heterogeneities in the lithosphere and the mantle from the analysis of $S$ and ScS seismogram envelopes by using our method.

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